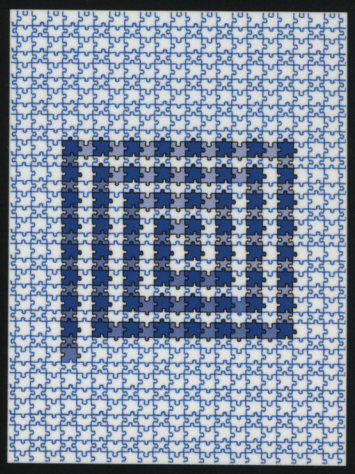
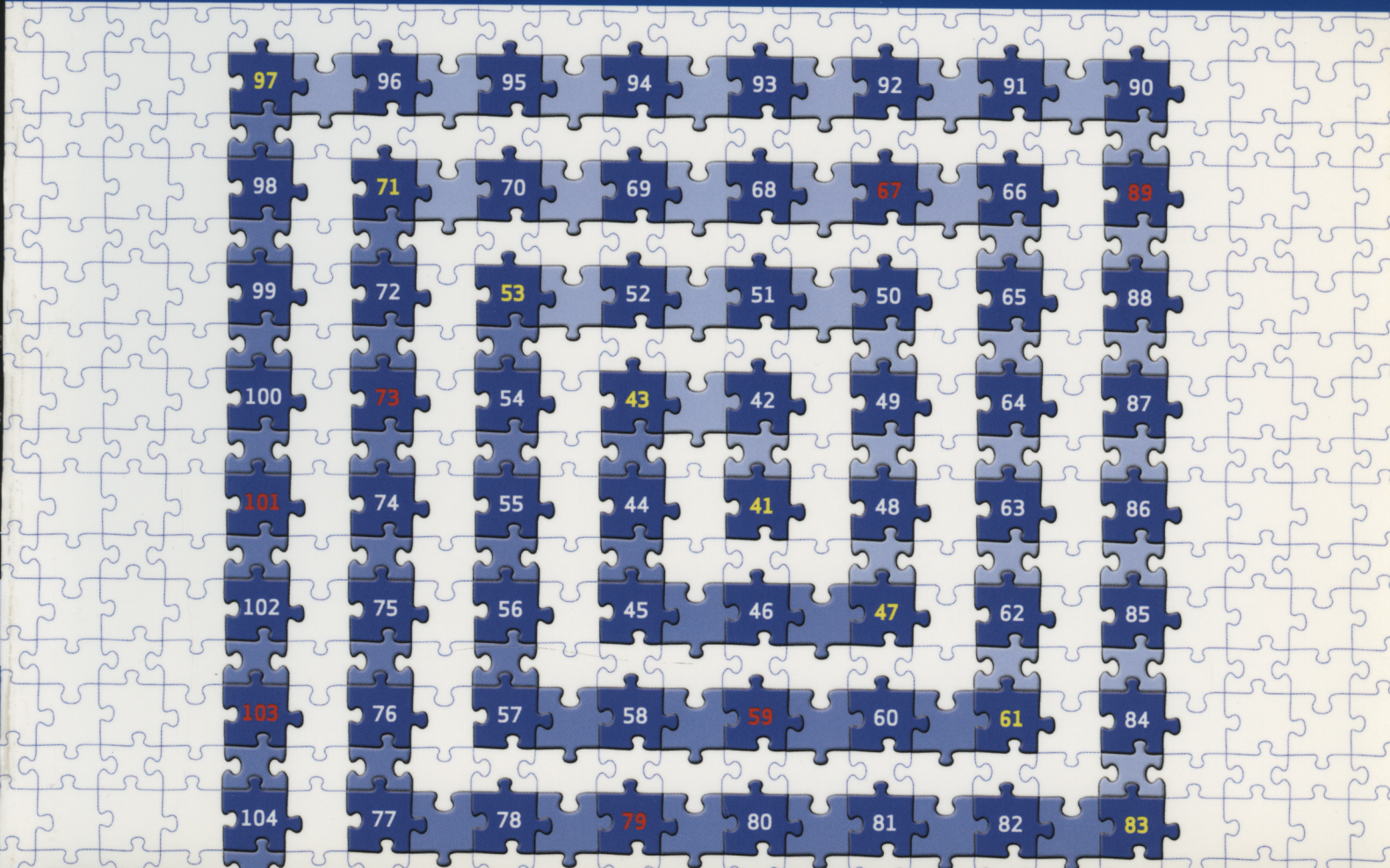
The Open University

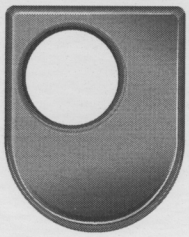
M381 Number Theory and Mathematical Logic



Number Theory **Unit 7**

Continued Fractions





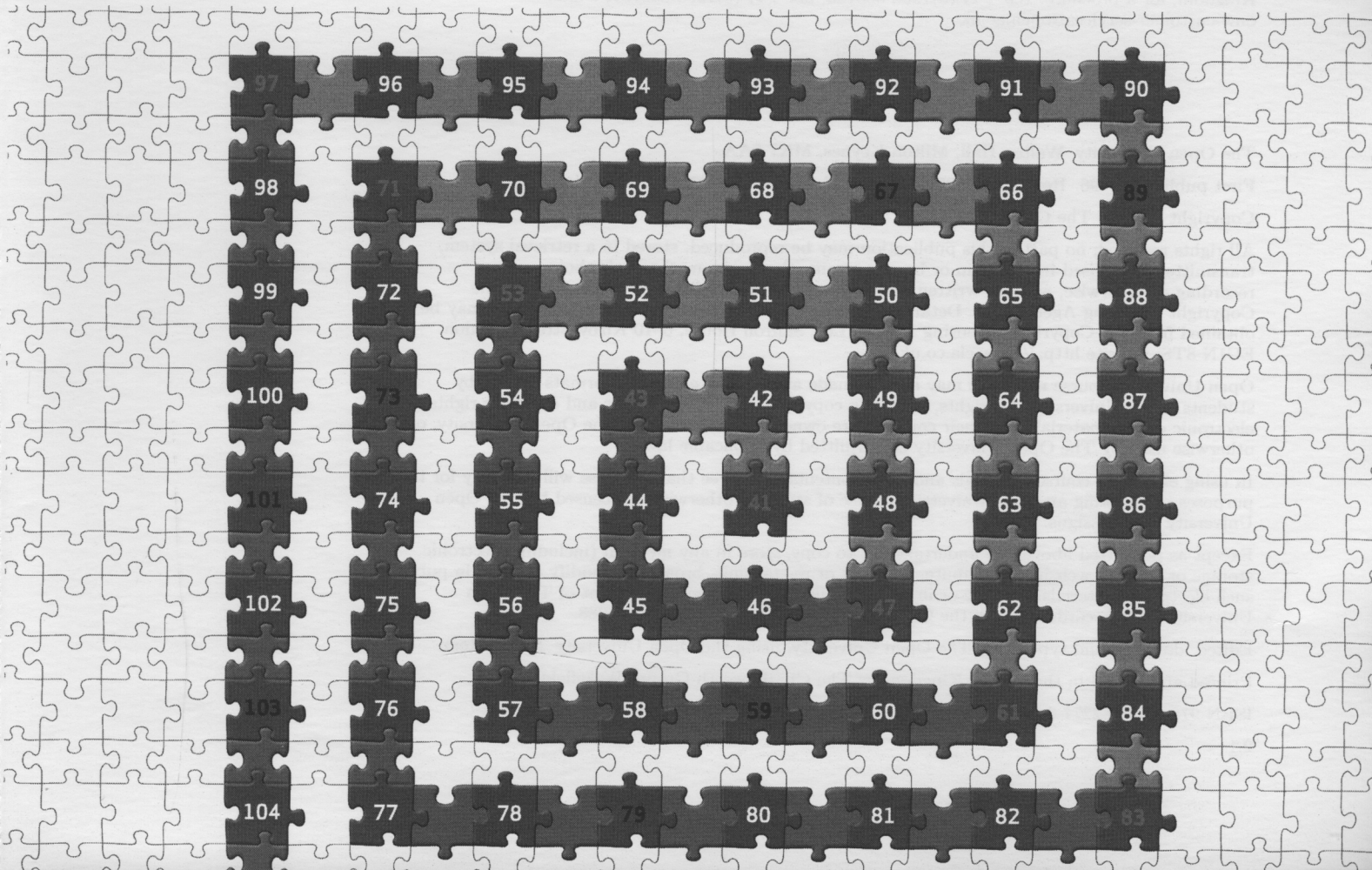
The Open University

M381 Number Theory and Mathematical Logic

Number Theory **Unit 7**

Continued Fractions

Prepared for the Course Team by Alan Best



The M381 Number Theory Course Team

The Number Theory half of the course was produced by the following team:

Alan Best	*Author*
Andrew Brown	*Course Team Chair* and *Academic Editor*
Roberta Cheriyan	*Course Manager*
Bob Coates	*Critical Reader*
Dick Crabbe	*Publishing Editor*
Janis Gilbert	*Graphic Artist*
Derek Goldrei	*Critical Reader*
Caroline Husher	*Graphic Designer*
John Taylor	*Graphic Artist*

with valuable assistance from:

CMPU	*Mathematics and Computing, Course Materials Production Unit*
John Bayliss	*Reader*
Elizabeth Best	*Reader*
Jeremy Gray	*History Reader*
Alison Neil	*Reader*

The external assessor was:

Alex Wilkie	*Reader in Mathematical Logic, University of Oxford*

This publication forms part of an Open University course. Details of this and other Open University courses can be obtained from the Student Registration and Enquiry Service, The Open University, PO Box 197, Milton Keynes, MK7 6BJ, United Kingdom: tel. +44 (0)870 300 6090, e-mail general-enquiries@open.ac.uk

Alternatively, you may visit the Open University website at http://www.open.ac.uk where you can learn more about the wide range of courses and packs offered at all levels by The Open University.

To purchase a selection of Open University course materials, visit http://www.ouw.co.uk, or contact Open University Worldwide, Michael Young Building, Walton Hall, Milton Keynes, MK7 6AA, United Kingdom, for a brochure: tel. +44 (0)1908 858793, fax +44 (0)1908 858787, e-mail ouw-customer-services@open.ac.uk

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1996. Reprinted 1997 and 2001. New edition 2007 with corrections.

Copyright © 1996 The Open University

All rights reserved; no part of this publication may be reproduced, stored in a retrieval system, transmitted or utilised in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without written permission from the publisher or a licence from the Copyright Licensing Agency Ltd. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd, Saffron House, 6–10 Kirby Street, London EC1N 8TS; website http://www.cla.co.uk.

Open University course materials may also be made available in electronic formats for use by students of the University. All rights, including copyright and related rights and database rights, in electronic course materials and their contents are owned by or licensed to The Open University, or otherwise used by The Open University as permitted by applicable law.

In using electronic course materials and their contents you agree that your use will be solely for the purposes of following an Open University course of study or otherwise as licensed by The Open University or its assigns.

Except as permitted above you undertake not to copy, store in any medium (including electronic storage or use in a website), distribute, transmit or re-transmit, broadcast, modify or show in public such electronic materials in whole or in part without the prior written consent of The Open University or in accordance with the Copyright, Designs and Patents Act 1988.

Edited, designed and typeset by The Open University, using the Open University TEX System.

Printed and bound in the United Kingdom by The Charlesworth Group, Wakefield.

ISBN 978 0 7492 2277 2

2.1

CONTENTS

Introduction 4

1 Finite Continued Fractions 7

1.1 The continued fraction of a rational number 7

1.2 The convergents of a continued fraction 11

1.3 Properties of the convergents 15

2 Infinite Continued Fractions 20

2.1 The convergence of a continued fraction 20

2.2 Periodic continued fractions 22

3 The Continued Fraction Algorithm 26

3.1 The continued fraction of an irrational 26

3.2 Using the Continued Fraction Algorithm 28

3.3 The continued fraction of $\sqrt{n}$ 30

4 Rational Approximations 32

4.1 Approximating irrationals by rationals 32

4.2 Best approximations 35

4.3 Rational approximations to π 37

Additional Exercises 40

Solutions to the Problems 42

Solutions to Additional Exercises 50

Index 56

INTRODUCTION

The quadratic equation $x^2 - x - 1 = 0$ is known to have roots

$$x = \frac{1 \pm \sqrt{5}}{2},$$

but suppose that we decide instead to solve this equation using an iterative method. The equation can be rearranged as

$$x = 1 + \frac{1}{x},$$

so we could find successive approximations, x_n, to a root by using the formula

$$x_{n+1} = 1 + \frac{1}{x_n}.$$

If we take $x_1 = 1$ as our initial approximation, the sequence begins as follows.

$$x_1 = 1$$

$$x_2 = 1 + \frac{1}{x_1} = 1 + \frac{1}{1} = \frac{2}{1}$$

$$x_3 = 1 + \frac{1}{x_2} = 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$$

$$x_4 = 1 + \frac{1}{x_3} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{5}{3}$$

$$x_5 = 1 + \frac{1}{x_4} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}} = \frac{8}{5}$$

It is interesting to see the Fibonacci numbers appearing as numerators and denominators of each x_n, and we might conjecture that $x_n = \frac{F_{n+1}}{F_n}$, for all positive integers n. This is certainly true when $n = 1$, and if we assume that $x_k = \frac{F_{k+1}}{F_k}$, then

$$x_{k+1} = 1 + \frac{1}{x_k} = 1 + \frac{F_k}{F_{k+1}} = \frac{F_{k+1} + F_k}{F_{k+1}} = \frac{F_{k+2}}{F_{k+1}}.$$

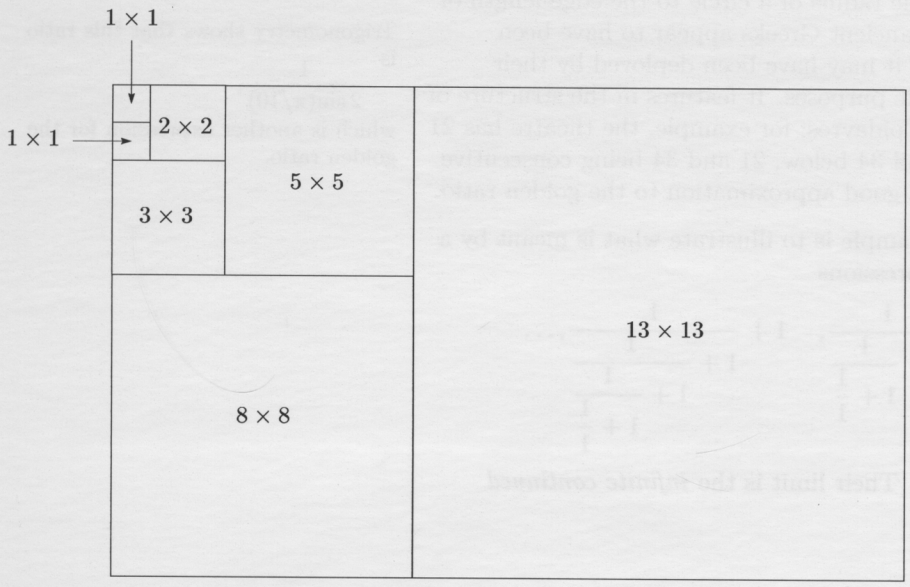
So Mathematical Induction confirms that it is true that $x_n = \frac{F_{n+1}}{F_n}$, for all $n \geq 1$.

The question remains as to whether or not the sequence $x_1, x_2, x_3, \ldots$ produced by this iterative process converges. The question is the same as asking whether the sequence $\frac{F_2}{F_1}, \frac{F_3}{F_2}, \frac{F_4}{F_3}, \ldots$ of ratios of successive Fibonacci numbers converges, and this problem has a nice geometric interpretation. Imagine squares with edge lengths 1, 1, 2, 3, 5, 8, 13, ... These can be joined, in order, to form a sequence of rectangles where, at each stage, the new rectangle is formed by adjoining the next square to the longer edge of the existing rectangle. This process is illustrated in the following figure. Now the shape of each rectangle, as determined by the ratio of the longer to the shorter side, is the ratio of successive Fibonacci numbers. So the question becomes: does this sequence of rectangles have a limiting shape?

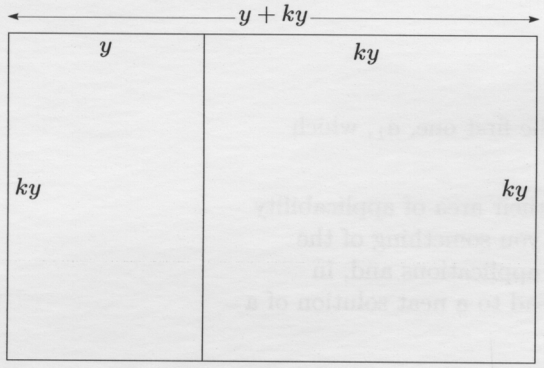
The roots of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We met the Fibonacci numbers in Section 5 of *Unit 2*.



The answer, as we shall see from the work of this unit, is yes. If, for the moment, we assume that the sequence has a limit k, say, then it is not too difficult to see what k must be. For sufficiently large rectangles the ratio of the lengths of edges is very close to k, so we could take these lengths to be (approximately) ky and y. To obtain the next rectangle we add on a square of side ky as shown below.



The ratio of the sides of this larger rectangle is also (approximately) k and so

$$\frac{y+ky}{ky} = k,$$

which, after cancelling the y, simplifies to $k^2 - k - 1 = 0$. Hence, if the sequence tends to a limit, it will be a root of the quadratic equation from which we started, $x^2 - x - 1 = 0$. The positive root is $k = \dfrac{1+\sqrt{5}}{2}$, a number which is known as the *golden ratio.*

The value of the golden ratio is approximately 1.6180.

The golden ratio was, along with π and $\sqrt{2}$, one of the first irrational numbers known to, and used by, the ancient Greeks, arising originally in the geometrical problem of dividing a line segment AB at a point C such that $\dfrac{AB}{AC} = \dfrac{AC}{CB}$, this common ratio turning out to be the golden ratio.

The golden ratio crops up all over mathematics and is particularly prominent in problems concerning regular plane and solid figures.

As one example, it is the ratio of the radius of a circle to the edge-length of an inscribed regular decagon. The ancient Greeks appear to have been endeared with the golden ratio and it may have been deployed by their sculptors and architects for aesthetic purposes. It features in the structure of the Parthenon and the theatre at Epidavros; for example, the theatre has 21 tiers of seats above the gangway and 34 below, 21 and 34 being consecutive Fibonacci numbers whose ratio is a good approximation to the golden ratio.

Trigonometry shows that this ratio is $\dfrac{1}{2\sin(\pi/10)}$, which is another expression for the golden ratio.

Our main purpose in giving this example is to illustrate what is meant by a continued fraction. Each of the expressions

$$1+\frac{1}{1},\quad 1+\cfrac{1}{1+\frac{1}{1}},\quad 1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}},\quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}},\ldots$$

is called a *finite continued fraction*. Their limit is the *infinite continued fraction*:

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$$

This is the simplest, and best-known example of a continued fraction. In the more general infinite continued fraction

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{\ddots}}}}$$

the a_i are positive integers, with the exception of the first one, a_1, which may be a positive or negative integer or zero.

The theory of continued fractions is extensive and their area of applicability very wide, so in this unit we can only hope to give you something of the flavour of them. We shall give a number of diverse applications and, in *Unit 8*, we shall see how they can be exploited to lead to a neat solution of a famous Diophantine equation.

We shall first explore finite continued fractions and, in so doing, lay the foundations for our subsequent investigations of infinite continued fractions. We shall discover that every infinite continued fraction converges to an irrational number and, conversely, that every irrational number can be expressed, in a unique way, as an infinite continued fraction.

We are all in the habit of writing down, and even manipulating, expressions such as π, $\log_e 3$, $\dfrac{1+\sqrt{5}}{2}$, etc., which represent irrational numbers. But when we come to practical calculations involving them we must deal with rational numbers. (Even today's computers are limited in how many decimal places of π they can handle.) So the question of finding good rational approximations to irrational numbers is an important one. In the final section of the unit, we see how the continued fraction of an irrational number provides us with 'best possible' rational approximations.

1 FINITE CONTINUED FRACTIONS

1.1 The continued fraction of a rational number

We start with the key definitions.

Definition 1.1 Finite Continued Fractions

Let a_1 be any real number and let $a_2, a_3, \ldots, a_n$ be any positive real numbers. The expression

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is a *finite continued fraction.* The numbers a_1, a_2, ..., a_n are the *partial quotients* of the continued fraction. When all the partial quotients are integers the continued fraction is said to be *simple.*

Continued fractions can be cumbersome objects and it is therefore advantageous to have a convenient symbol to display them. Any continued fraction is determined by its partial quotients and so we write the finite continued fraction in the definition as

$$[a_1, a_2, a_3, \ldots, a_n].$$

For example,

$$[1,2,5] = 1 + \cfrac{1}{2 + \cfrac{1}{5}}$$

and the expression on the right simplifies to the rational number $\frac{16}{11}$ which is the *value* of the continued fraction. Similarly the value of $[-3,1,2,4]$ can be calculated as follows.

$$[-3,1,2,4] = -3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4}}} = -3 + \cfrac{1}{1 + \frac{4}{9}} = -3 + \frac{9}{13} = -\frac{30}{13}$$

Note that formally we think of the finite continued fraction and its value as being distinct. The continued fraction is an arithmetic representation of a number (its value). However we shall occasionally drop this distinction and write, for example, $x = [-3,1,2,4]$ to mean that x is the value of this continued fraction.

Problem 1.1

Determine the rational numbers represented by the following finite continued fractions.

(a) $[2,3,4]$ (b) $[0,2,2,2]$

Looking at the first step in the calculations in the examples above, note that

$$[1,2,5] = 1 + \frac{1}{[2,5]} \quad \text{and} \quad [-3,1,2,4] = -3 + \frac{1}{[1,2,4]}.$$

In general, from the way the finite continued fraction is defined, we have the following property.

First continued fraction identity

$$[a_1, a_2, a_3, \ldots, a_n] = a_1 + \frac{1}{[a_2, a_3, \ldots, a_n]}$$

In our work we are going to be concerned almost exclusively with simple continued fractions; that is, ones in which all the partial quotients are integers. There will, however, be isolated occasions when it proves convenient to use continued fractions with real partial quotients for the purpose of providing general proofs, though the results will subsequently only be applied to cases involving simple continued fractions.

In the text, for ease of notation, we shall use the abbreviation FCF to stand for a *simple finite continued fraction.*

It is readily seen from the definition that the value of any FCF is a rational number. As it happens the converse also holds: any rational number can be expressed as an FCF. The Euclidean Algorithm provides a means of determining an FCF for a given rational number. We illustrate this method in the following example.

Example 1.1

Determine an FCF for $\frac{225}{157}$.

Dividing 225 by 157, the Euclidean Algorithm gives the sequence of equations on the left below. Each of these is written in an alternative form on the right.

$$\begin{aligned} 225 &= 1 \times 157 + 68 \quad \text{or} \quad \frac{225}{157} = 1 + \frac{68}{157} \\ 157 &= 2 \times 68 + 21 \qquad \frac{157}{68} = 2 + \frac{21}{68} \\ 68 &= 3 \times 21 + 5 \qquad \frac{68}{21} = 3 + \frac{5}{21} \\ 21 &= 4 \times 5 + 1 \qquad \frac{21}{5} = 4 + \frac{1}{5} \\ 5 &= 5 \times 1 \qquad \frac{5}{1} = 5 \end{aligned}$$

From the system of equations on the right, by continually using the next equation to substitute for the existing fraction, we obtain the following.

$$\begin{aligned} \frac{225}{157} &= 1 + \frac{68}{157} = 1 + \frac{1}{\frac{157}{68}} \\ &= 1 + \frac{1}{2 + \frac{21}{68}} = 1 + \frac{1}{2 + \frac{1}{\frac{68}{21}}} \\ &= 1 + \frac{1}{2 + \frac{1}{3 + \frac{5}{21}}} = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{\frac{21}{5}}}} \\ &= 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}} \end{aligned}$$

In other words, $\dfrac{225}{157} = [1, 2, 3, 4, 5]$, the partial quotients of the FCF being, in order, precisely the quotients as they arise in the Euclidean Algorithm. ♦

Problem 1.2

Determine an FCF for each of the following rational numbers.

(a) $\dfrac{11}{7}$ (b) $\dfrac{43}{19}$

This way of using the Euclidean Algorithm can be deployed to obtain an FCF for *any* rational number. Let us formalize this before trying more examples.

> ***Theorem 1.1 Equivalence of FCFs and rational numbers***
>
> Every simple finite continued fraction has a rational value and, conversely, every rational number can be expressed as a simple finite continued fraction.

Proof of Theorem 1.1

We prove that the value of any FCF is rational by Mathematical Induction on the number of partial quotients.

The induction is started by observing that any FCF with a single partial quotient, $[a_1]$, is an integer and therefore rational.

By definition $[a_1] = a_1$.

For the induction step, suppose that the value of any FCF with k partial quotients is rational and consider the FCF $[a_1, a_2, a_3, \ldots, a_{k+1}]$ with $k+1$ partial quotients. We now use the first continued fraction identity:

$$[a_1, a_2, a_3, \ldots, a_{k+1}] = a_1 + \frac{1}{[a_2, a_3, \ldots, a_{k+1}]}.$$

Now a_1, being an integer, is rational. Furthermore, by the induction hypothesis, $[a_2, a_3, \ldots, a_{k+1}]$ is rational and hence so is its reciprocal. It follows that $[a_1, a_2, a_3, \ldots, a_{k+1}]$, being the sum of two rationals, is itself rational.

As $a_2 > 0$, by definition, $[a_2, a_3, \ldots, a_{k+1}] \neq 0$.

Mathematical Induction therefore confirms that the value of any FCF is a rational number.

Conversely, let $\dfrac{a}{b}$ be any rational number. By the Euclidean Algorithm:

When we write $\dfrac{a}{b}$ for a rational number we assume that a, b are integers with b positive and $\gcd(a, b) = 1$.

$$a = q_1 b + r_1, \quad 0 < r_1 < b \quad \text{or} \quad \frac{a}{b} = q_1 + \frac{r_1}{b}$$

$$b = q_2 r_1 + r_2, \quad 0 < r_2 < r_1 \quad \text{or} \quad \frac{b}{r_1} = q_2 + \frac{r_2}{r_1}$$

$$r_1 = q_3 r_2 + r_3, \quad 0 < r_3 < r_2 \quad \text{or} \quad \frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$r_{n-3} = q_{n-1} r_{n-2} + r_{n-1}, \quad 0 < r_{n-1} < r_{n-2} \quad \text{or} \quad \frac{r_{n-3}}{r_{n-2}} = q_{n-1} + \frac{r_{n-1}}{r_{n-2}}$$

$$r_{n-2} = q_n r_{n-1} + 0 \quad \text{or} \quad \frac{r_{n-2}}{r_{n-1}} = q_n$$

The fact that the non-zero remainders decrease guarantees that the algorithm stops after a finite number of steps.

From the right-hand system of equations we obtain

$$\frac{a}{b} = q_1 + \frac{1}{\frac{b}{r_1}} = q_1 + \frac{1}{q_2 + \frac{1}{\frac{r_1}{r_2}}} = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{\frac{r_2}{r_3}}}} = \cdots$$

$$= q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\ddots \cfrac{}{q_{n-1} + \cfrac{1}{q_n}}}}},$$

which shows that $\frac{a}{b}$ has the FCF $[q_1, q_2, .., q_n]$. ■

Problem 1.3

Obtain FCFs for the following rational numbers.

(a) $\frac{65}{23}$ (b) $\frac{23}{53}$ (c) $\frac{-94}{13}$

The FCF of a rational number is not unique. In fact each rational number has exactly two FCFs. The ambiguity arises because of the freedom in choosing whether or not to terminate the FCF with partial quotient 1. To substantiate that remark, look again at Example 1.1 where we saw that $\frac{225}{157} = [1, 2, 3, 4, 5]$. The final partial quotient, 5, can be written as $4 + \frac{1}{1}$:

$$\cdots + \frac{1}{4 + \frac{1}{5}} = \cdots + \frac{1}{4 + \frac{1}{4 + \frac{1}{1}}}.$$

Consequently we have an alternative form for the FCF

$$\frac{225}{157} = [1, 2, 3, 4, 4, 1].$$

Similarly, from Problem 1.3 part (a) above

$$\frac{65}{23} = [2, 1, 4, 1, 3] = [2, 1, 4, 1, 2, 1],$$

the alternative being reached by writing the final partial quotient 3 as $2 + \frac{1}{1}$. In general, if $a_n > 1$,

$$[a_1, a_2, \ldots, a_{n-1}, a_n] = [a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1],$$

since $a_n = (a_n - 1) + \frac{1}{1}$.

Problem 1.4

Write each of the following as an FCF with an even number of partial quotients.

(a) $[1, 2, 3]$ (b) $[-4, 3, 1, 6, 1]$ (c) $[0]$

If an FCF has a final partial quotient of 1 then, in the way illustrated in Problem 1.4 part (b), the final 1 can be absorbed into the previous partial quotient. If we choose always to absorb such 1s then the ambiguity in the FCF is removed and every rational number has a unique FCF whose final quotient is other than 1. (There is one exceptional number. The integer 1 has FCF [1], where the final partial quotient 1 cannot be absorbed into the previous one because there is no previous one. The two FCFs of 1 are [1] and

[0, 1], both of which end in 1.) We shall not give a proof of this uniqueness here for, in Section 3, we shall prove a general result concerning the uniqueness of infinite continued fractions which can be applied to the finite case.

1.2 The convergents of a continued fraction

Let us turn to the question of evaluating FCFs. First we need to introduce some further terminology.

Definition 1.2 Convergents

For each $k = 1, 2, \ldots, n$ we define the *kth convergent*, C_k, of the FCF $[a_1, a_2, \ldots, a_n]$ to be the value of $[a_1, a_2, \ldots, a_k]$.

Notice that the final convergent C_n is the value of the FCF itself.

The reason for the name *convergent* will become apparent in due course.

As an illustration of the definition consider $[1, 2, 4, 3, 2]$. It has five convergents.

$$C_1 = [1] = 1$$

$$C_2 = [1, 2] = 1 + \frac{1}{2} = \frac{3}{2}$$

$$C_3 = [1, 2, 4] = 1 + \frac{1}{2 + \frac{1}{4}} = \frac{13}{9}$$

$$C_4 = [1, 2, 4, 3] = 1 + \frac{1}{2 + \frac{1}{4 + \frac{1}{3}}} = \frac{42}{29}$$

$$C_5 = [1, 2, 4, 3, 2] = 1 + \frac{1}{2 + \frac{1}{4 + \frac{1}{3 + \frac{1}{2}}}} = \frac{97}{67}$$

Although, for example, we have, written $C_4 = [1, 2, 4, 3]$ the convergent C_4 is really the value of this expression, namely $\frac{42}{29}$.

At first sight it appears that the only way of evaluating an FCF is the logical one of starting with the final partial quotient and working our way upward through the expression, determining the various fractions along the way. Fortunately this is not so. We can evaluate the convergents C_1, C_2, ... directly, obtaining each one from the convergents which have gone before it until we reach the one we want. Our next result will show how this is done.

A key step in the proof of the coming theorem is going to be the following identity.

Second continued fraction identity

$$[a_1, a_2, \ldots, a_k, a_{k+1}] = [a_1, a_2, \ldots, a_{k-1}, a_k + \frac{1}{a_{k+1}}]$$

This is obtained by absorbing the final partial quotient of the left-hand side into the penultimate one. This enables us to write a continued fraction with $k + 1$ partial quotients in an alternative form with just k partial quotients. However, the expression on the right-hand side of this equality is not a simple continued fraction because, in general, $a_k + \frac{1}{a_{k+1}}$ is not an integer. For this reason we shall state, and prove, our theorem for the case of the

$a_k + \frac{1}{a_{k+1}}$ will be an integer only when $a_{k+1} = 1$.

general continued fraction, although thereafter our applications will be restricted to the special case where the continued fraction is simple.

Theorem 1.2 The value of a finite continued fraction

Let $[a_1, a_2, \ldots, a_n]$ be any finite continued fraction. Define the *numerators* $p_1, p_2, p_3, \ldots, p_n$ and the *denominators* $q_1, q_2, q_3, \ldots, q_n$ of this continued fraction by:

$$\begin{array}{llll} p_1 = a_1, & p_2 = a_1a_2 + 1; & p_k = a_kp_{k-1} + p_{k-2}, & 3 \leq k \leq n; \\ q_1 = 1, & q_2 = a_2; & q_k = a_kq_{k-1} + q_{k-2}, & 3 \leq k \leq n. \end{array}$$

Then the value of $[a_1, a_2, \ldots, a_n]$ is $\dfrac{p_n}{q_n}$.

Before we tackle the proof, let us get a feel for the formulae involved here by working through some examples.

Example 1.2

Use the formulae of Theorem 1.2 to determine the value of $[3, 3, 1, 2, 5]$.

The first two numerators are

$$p_1 = a_1 = 3 \quad \text{and} \quad p_2 = a_1a_2 + 1 = 3 \times 3 + 1 = 10.$$

Subsequent numerators are given by the iterative formula $p_k = a_kp_{k-1} + p_{k-2}$.

$$p_3 = 1 \times p_2 + p_1 = 1 \times 10 + 3 = 13$$
$$p_4 = 2 \times p_3 + p_2 = 2 \times 13 + 10 = 36$$
$$p_5 = 5 \times p_4 + p_3 = 5 \times 36 + 13 = 193$$

The first two denominators are

$$q_1 = 1 \quad \text{and} \quad q_2 = a_2 = 3.$$

Subsequent denominators are given by the same iterative formula as for the numerators, $q_k = a_kq_{k-1} + q_{k-2}$.

$$q_3 = 1 \times q_2 + q_1 = 1 \times 3 + 1 = 4$$
$$q_4 = 2 \times q_3 + q_2 = 2 \times 4 + 3 = 11$$
$$q_5 = 5 \times q_4 + q_3 = 5 \times 11 + 4 = 59$$

The value of $[3, 3, 1, 2, 5]$ is, according to the theorem, $\dfrac{p_5}{q_5} = \dfrac{193}{59}$. ♦

Notice that the calculations in the above example give us more than just the value of the FCF; we have determined the values of all the convergents along the way. For example, the theorem tells us that the third convergent $[3, 3, 1]$ has value $\dfrac{p_3}{q_3} = \dfrac{13}{4}$. Indeed the convergents are precisely $\dfrac{p_1}{q_1}, \dfrac{p_2}{q_2}, \dfrac{p_3}{q_3}, \dfrac{p_4}{q_4}$ and $\dfrac{p_5}{q_5}$. That is, $\dfrac{3}{1}, \dfrac{10}{3}, \dfrac{13}{4}, \dfrac{36}{11}$ and $\dfrac{193}{59}$, the last of which is the value of the FCF itself.

This example illustrates an immediate corollary to the above theorem.

Corollary to Theorem 1.2

Let $[a_1, a_2, \ldots, a_n]$ be any finite continued fraction and the numerators $p_1, p_2, \ldots p_n$ and denominators $q_1, q_2, \ldots, q_n$ be as defined in the theorem. Then its convergents are

$$C_k = \frac{p_k}{q_k}, \quad \text{for } k = 1, 2, \ldots, n.$$

Problem 1.5

Use the corollary to Theorem 1.2 to determine the convergents of the following FCFs.

(a) $[2, 3, 4, 1, 2]$ (b) $[-2, 3, 1, 6]$

In the next example we suggest a tabular method for calculating the successive numerators and denominators. This method may help you avoid arithmetic slips and might well assist you in remembering these important formulae. However, after this subsection we shall not display tables again; we shall just use the formulae to write down the convergents.

Example 1.3

Determine the rational number represented by [4, 2, 1, 3, 1, 2, 4].

In the table we set out four rows labelled k, p_k, a_k and q_k. Since this continued fraction has seven partial quotients, k takes the values from 1 to 7. The a_k are the given partial quotients and so the entries for this row can be displayed immediately. The p_k and q_k rows are started by entering the first two values for each using the formulae $p_1 = a_1$, $p_2 = a_1a_2 + 1$, $q_1 = 1$ and $q_2 = a_2$.

k	1	2	3	4	5	6	7
p_k	4	9					
a_k	4	2	1	3	1	2	4
q_k	1	2					

To complete the table we move along the a_k row filling in entries above and below each a_k. For the entry above a_k (i.e. p_k), we multiply a_k by the entry diagonally above and to the left, and add to this product the entry to the left of the latter (Figure 1.1). So, for example, $p_3 = 1 \times 9 + 4 = 13$ and $p_4 = 3 \times 13 + 9 = 48$.

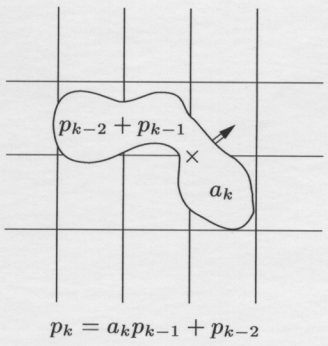


Figure 1.1

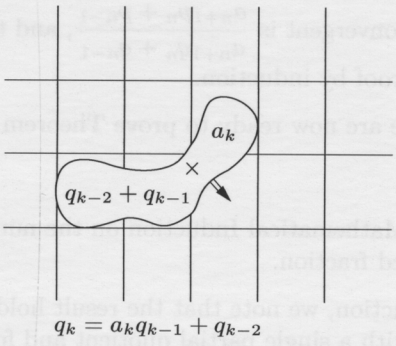


Figure 1.2

The same procedure applies for the q_k. We multiply a_k by the entry diagonally below and to the left and add on to this product the entry to the left of the latter (Figure 1.2). So, $q_3 = 1 \times 2 + 1 = 3$, and so on.

The complete table is

k	1	2	3	4	5	6	7
p_k	4	9	13	48	61	170	741
a_k	4	2	1	3	1	2	4
q_k	1	2	3	11	14	39	170

and the convergents are $\frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}$ and $\frac{741}{170}$, the final one being the value of the FCF. ♦

Problem 1.6

Use the tabular method to determine the convergents of each of the following FCFs.

(a) $[-4, 1, 5, 1, 6]$ (b) $[1, 1, 2, 1, 2, 1, 2]$

Problem 1.7

In Problem 1.5 part (a) we determined the convergents of $[2, 3, 4, 1, 2]$. Now determine the convergents of the following FCFs.

(a) $[2, 3, 4, 1, 2, 2]$ (b) $[2, 3, 4, 1, 2, 100]$

Problem 1.7 highlights another property of continued fractions. The convergents of $[2, 3, 4, 1, 2]$ are $\frac{2}{1}, \frac{7}{3}, \frac{30}{13}, \frac{37}{16}$ and $\frac{104}{45}$, and any continued fraction which begins $[2, 3, 4, 1, 2, \ldots]$ will have these five as its first, second, third, fourth and fifth convergents no matter what follows. There is nothing special about $[2, 3, 4, 1, 2]$ here. If $[a_1, a_2, \ldots, a_n]$ is any continued fraction with convergents $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}$, then the continued fraction obtained by adding a further partial quotient, $[a_1, a_2, \ldots, a_n, a_{n+1}]$, has the same convergents $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}$ but with one extra final one, which will be equal to the value of the (new) continued fraction itself. According to Theorem 1.2, this final convergent is $\frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}$, and this is what we have to establish for a proof by induction.

With this observation we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2

We prove the result by Mathematical Induction on the number n of partial quotients in the continued fraction.

As the basis for the induction, we note that the result holds for any continued fraction $[a_1]$ with a single partial quotient and for any continued fraction $[a_1, a_2]$ with two partial quotients since

$$[a_1] = a_1 \quad \text{and} \quad \frac{p_1}{q_1} = \frac{a_1}{1} = a_1$$

and

$$[a_1, a_2] = a_1 + \frac{1}{a_2} \quad \text{and} \quad \frac{p_2}{q_2} = \frac{a_1 a_2 + 1}{a_2} = a_1 + \frac{1}{a_2}.$$

Now suppose that the result holds for any continued fraction with $k \geq 2$ partial quotients and consider the finite continued fraction

$$[a_1, a_2, \ldots, a_{k+1}]$$

with $k+1$ partial quotients. The numerators of this continued fraction are $p_1, p_2, \ldots, p_k$ and p_{k+1}, and the denominators are $q_1, q_2, \ldots, q_k$ and q_{k+1}. Our task is to show that the value of the continued fraction is $\dfrac{p_{k+1}}{q_{k+1}}$.

Now, by the second continued fraction identity,

$$[a_1, a_2, \ldots, a_{k+1}] = [a_1, a_2, \ldots, a_{k-1}, a'_k], \text{ where } a'_k = a_k + \frac{1}{a_{k+1}}.$$

Look at the right-hand continued fraction. Its numerators are

$$p_1, p_2, \ldots, p_{k-1} \quad \text{and} \quad p'_k = \left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}.$$

Similarly its denominators are

$$q_1, q_2, \ldots, q_{k-1} \quad \text{and} \quad q'_k = \left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}.$$

Moreover, as it has just k partial quotients, the induction hypothesis tells us that its value is $\dfrac{p'_k}{q'_k}$. That is,

$$\begin{aligned}
[a_1, a_2, \ldots, a_{k+1}] &= \frac{p'_k}{q'_k} \\
&= \frac{\left(a_k + \dfrac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \dfrac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}} \\
&= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\
&= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}.
\end{aligned}$$

This completes the induction step and therefore, by Mathematical Induction, the result is true for a continued fraction with any number of partial quotients. ■

1.3 Properties of the convergents

The convergents of an FCF have a number of curious properties. For example, let us look again at the sequence of convergents of $[4, 2, 1, 3, 1, 2, 4]$ which we found in Example 1.3.

p_k	4	9	13	48	61	170	741
q_k	1	2	3	11	14	39	170

The first thing you may have noticed is that, for each k, the numbers p_k and q_k are relatively prime. There is also something curious about the 'cross-products' of consecutive convergents. For example, for $\dfrac{9}{2}$ and $\dfrac{13}{3}$ we have $9 \times 3 = 27$ and $13 \times 2 = 26$. They differ by 1, something which happens with every pair of consecutive convergents in this example. You might like to check that $170 \times 14 - 61 \times 39 = 1$. This is a general property of the convergents of an FCF.

Theorem 1.3 Properties of the convergents

Let $[a_1, a_2, \ldots, a_n]$ be any FCF, and let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ be its numerators and denominators as defined in Theorem 1.2. Then we have the following properties.

(a) $p_k q_{k-1} - p_{k-1} q_k = (-1)^k$; that is,

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^k}{q_k q_{k-1}}, \quad \text{for } 2 \leq k \leq n.$$

(b) $p_k q_{k-2} - p_{k-2} q_k = (-1)^{k-1} a_k$; that is,

$$\frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}}, \quad \text{for } 3 \leq k \leq n.$$

(c) p_k and q_k are relatively prime for $k = 1, 2, \ldots, n$.

Theorem 1.2 told us that the convergents are $\frac{p_k}{q_k}$. Property (c) confirms that these numbers are rationals, written in their lowest terms.

Proof of Theorem 1.3

(a) We use Mathematical Induction to prove that, for any integer $r \geq 2$ the numerators p_{r-1} and p_r and the denominators q_{r-1} and q_r of the FCF $[a_1, a_2, \ldots, a_r]$ satisfy

$$p_r q_{r-1} - p_{r-1} q_r = (-1)^r.$$

The result of property (a) will follow by letting r range from 2 to n

For the case $r = 2$, $p_1 = a_1$, $p_2 = a_1 a_2 + 1$, $q_1 = 1$ and $q_2 = a_2$ so that

$$p_2 q_1 - p_1 q_2 = (a_1 a_2 + 1) \times 1 - a_1 a_2 = 1 = (-1)^2.$$

This gives the basis for the induction.

For the induction step, assume that it is true for $r = k$, so that, for the FCF $[a_1, a_2, \ldots, a_k]$, we have that $p_k q_{k-1} - p_{k-1} q_k = (-1)^k$, and consider $[a_1, a_2, \ldots, a_k, a_{k+1}]$. Its final numerator and denominator are, according to Theorem 1.2,

$$p_{k+1} = a_{k+1} p_k + p_{k-1} \quad \text{and} \quad q_{k+1} = a_{k+1} q_k + q_{k-1}.$$

Therefore

$$\begin{aligned} p_{k+1} q_k - p_k q_{k+1} &= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\ &= p_{k-1} q_k - p_k q_{k-1} \\ &= (-1)(p_k q_{k-1} - p_{k-1} q_k) \\ &= (-1)(-1)^k, \quad \text{by the induction hypothesis,} \\ &= (-1)^{k+1}, \end{aligned}$$

showing property (a) to be true when $r = k + 1$. This completes the proof.

(b) This is a simple consequence of property (a).

$$\begin{aligned} p_k q_{k-2} - p_{k-2} q_k &= (a_k p_{k-1} + p_{k-2}) q_{k-2} - p_{k-2} (a_k q_{k-1} + q_{k-2}) \\ &= a_k (p_{k-1} q_{k-2} - p_{k-2} q_{k-1}) \\ &= a_k (-1)^{k-1} \end{aligned}$$

(c) This also follows from property (a).

If $d = \gcd(p_k, q_k)$ then $p_k q_{k-1} - p_{k-1} q_k$, being an integer combination of p_k and q_k, is a multiple of d. Therefore, by property (a), d divides $(-1)^k$, which implies $d = 1$. ■

Let us pause at this point to look at one application of FCFs. The result of Theorem 1.3 leads to a neat way of solving linear Diophantine equations.

Example 1.4

Find the solution of the linear Diophantine equation $13x + 59y = 10$ in which x takes its smallest positive value.

We begin by finding the FCF of $\frac{59}{13}$.

$$59 = 4 \times 13 + 7$$
$$13 = 1 \times 7 + 6$$
$$7 = 1 \times 6 + 1$$
$$6 = 6 \times 1$$

So $\frac{59}{13} = [4, 1, 1, 6]$, and its convergents are $\frac{4}{1}, \frac{5}{1}, \frac{5+4}{1+1} = \frac{9}{2}$ and $\frac{6 \times 9 + 5}{6 \times 2 + 1} = \frac{59}{13}$. From the final two convergents, (as Theorem 1.3 guaranteed would occur), we observe that

$$59 \times 2 - 9 \times 13 = 1.$$

Multiplying through by 10 and rearranging gives

$$13 \times (-90) + 59 \times 20 = 10,$$

showing that $x = -90$, $y = 20$ is one solution of $13x + 59y = 10$. The general solution is therefore

$$x = -90 + 59t, \quad y = 20 - 13t, \quad t \in \mathbb{Z}.$$

For the smallest positive value that x can have in a solution take $t = 2$, to give the solution $x = 28$, $y = -6$. ♦

Problem 1.8

By determining the convergents of $\frac{48}{19}$, find the general solution of the linear Diophantine equation $48x + 19y = 4$.

We could equally well have used the convergents of $\frac{19}{48}$ but taking the larger number on the top is more efficient.

Returning to Theorem 1.3, the alternative formulations of properties (a) and (b) give important information about the convergents of an FCF.
Property (a) tells about the difference between successive convergents:

$$C_k - C_{k-1} = \frac{(-1)^k}{q_k q_{k-1}}, \quad \text{for } k = 2, 3, \ldots, n. \qquad (*)$$

Remember that $C_k = \frac{p_k}{q_k}$.

Now the denominators q_k satisfy

$$q_1 \leq q_2 < q_3 < \cdots < q_n.$$

We ask you to verify this in Problem 1.10

Therefore the right-hand side of equation $(*)$ becomes smaller as k increases. That is, the convergents are coming closer together.

Property (b), on the other hand, tells about the difference between convergents two apart:

$$C_k - C_{k-2} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}}, \quad \text{for } k = 3, 4, \ldots, n. \qquad (**)$$

We shall use both these facts in the proof of the following theorem concerning the relative sizes of convergents. For ease of notation we refer to the convergents $C_2, C_4, C_6, \ldots$, which have even subscripts, as the *even convergents* whilst $C_1, C_3, C_5, \ldots$ are called the *odd convergents.*

Theorem 1.4 The relative sizes of convergents

For any FCF the odd convergents form a strictly increasing sequence $C_1 < C_3 < C_5 < \cdots$, while the even convergents form a strictly decreasing sequence $C_2 > C_4 > C_6 > \cdots$.

Every even convergent is greater than every odd convergent.

We can illustrate pictorially what the theorem is claiming by imagining a plot of the points C_r against r.

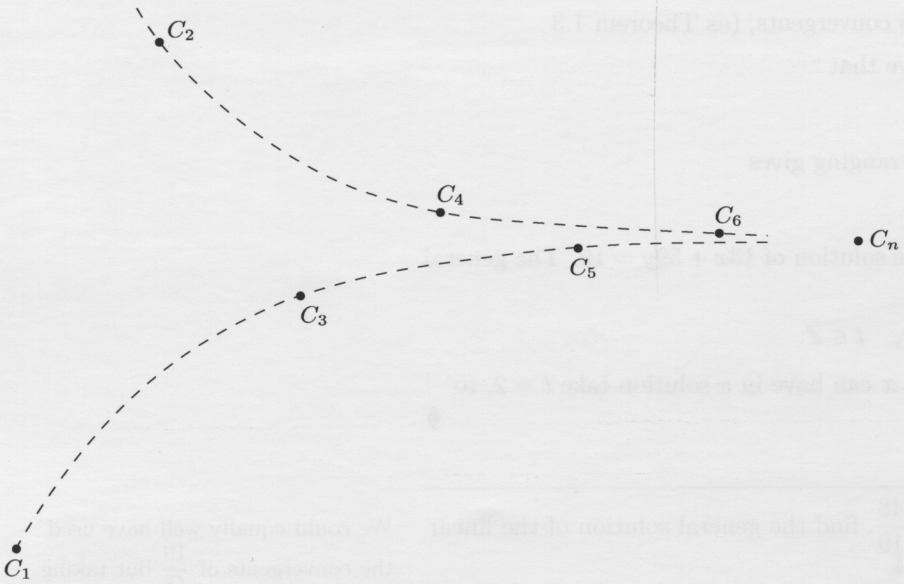


Figure 1.3 The convergents of an FCF

The final convergent C_n, which is the value of the FCF, may be either an even or an odd convergent. The last part of the theorem tells us that whichever it is, C_n is at least as large as every odd convergent and no larger than any even convergent.

Proof of Theorem 1.4

As $a_k > 0$ (for $k \geq 2$) and $q_k > 0$ (for $k \geq 1$) we note that, for $k \geq 3$,

$$\frac{(-1)^{k-1}a_k}{q_k q_{k-2}}$$

has the same sign as $(-1)^{k-1}$ and so will be positive when k is odd and negative when k is even.

Putting $k = 2r + 1$ in equation $(**)$ produces

$$C_{2r+1} - C_{2r-1} > 0, \quad \text{for } r \geq 1,$$

which confirms that the odd convergents are strictly increasing.

Putting $k = 2r$ in the same equation gives

$$C_{2r} - C_{2r-2} < 0, \quad \text{for } r > 1,$$

which confirms that the even convergents are strictly decreasing.

It remains to show that every odd convergent is less than every even convergent, and for that we use equation $(*)$ which tells us that $C_k - C_{k-1}$ takes the same sign as $(-1)^k$, for $k = 2, 3, \ldots, n$. Putting $k = 2r$ gives

$$C_{2r} - C_{2r-1} > 0, \quad \text{for } r \geq 1,$$

and putting $k = 2r + 1$ gives

$$C_{2r+1} - C_{2r} < 0, \quad \text{for } r \geq 1.$$

Hence the even convergent C_{2r} is larger than each of the adjacent odd convergents C_{2r-1} and C_{2r+1}.

Now consider any even convergent C_{2s} and any odd convergent C_{2t+1}.

If $s \geq t$ then $C_{2s} > C_{2s+1} \geq C_{2t+1}$, the second inequality utilizing the fact that the odd convergents increase.

If $s < t$ then $C_{2s} > C_{2t} > C_{2t+1}$, the first inequality utilizing the fact that the even convergents decrease.

In each case $C_{2s} > C_{2t+1}$, as required. ■

Problem 1.9

Determine the convergents of [2, 1, 1, 2, 1, 1, 2] and confirm that they satisfy Theorem 1.4.

Problem 1.10

Show that, for any FCF, the denominators q_k are increasing, and are strictly increasing with the possible exception of $q_1 = q_2$. Show further that, for any FCF, $q_k \geq k$ for all $k \geq 5$.

2 INFINITE CONTINUED FRACTIONS

2.1 The convergence of a continued fraction

In the introduction to the unit we built up a sequence of FCFs

$$[1],\ [1,1],\ [1,1,1],\ [1,1,1,1],\ldots$$

whose rational values give better and better approximations to the golden ratio, $\dfrac{1+\sqrt{5}}{2}$. However, the value of any FCF is rational, but the golden ratio is irrational, and so no matter how far we go along this sequence of FCFs we can never reach the golden ratio itself. What is of interest is the limit of this sequence of rationals. This suggests that we might explore the idea of the unending continued fraction $[1,1,1,\ldots]$.

This is our first example of a simple *infinite* continued fraction.

Definition 2.1 Infinite Continued Fractions.

Let a_1 be any real number and let $a_2, a_3, \ldots, a_n, \ldots$, be any positive real numbers. The expression

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \cfrac{}{a_n + \cfrac{1}{\ddots}}}}}$$

is an *infinite continued fraction.* We write this continued fraction in terms of its partial quotients as $[a_1, a_2, a_3, \ldots]$. Its convergents are the finite continued fractions

$$C_1 = [a_1],\ C_2 = [a_1, a_2],\ C_3 = [a_1, a_2, a_3], \ldots.$$

When all the partial quotients are integers, the continued fraction is said to be *simple.* We shall use the abbreviation ICF for a simple infinite continued fraction.

In this subsection we shall explore basic properties of infinite continued fractions. Our ultimate interest will lie with simple infinite continued fractions although, as in the previous section, we shall have occasion to allow the partial quotients to be non-integers for the purpose of providing general proofs.

What exactly does an infinite continued fraction represent? Whereas calculating the value of a finite continued fraction involves only the normal operations of arithmetic, evaluation of an ICF is an infinite process and as such requires a different approach. There is a strong resemblance to the convergence question for infinite series. The sum of the infinite series

$$\sum_{r=1}^{\infty} x_r = x_1 + x_2 + x_3 + \cdots$$

is defined to be the limit of the sequence

$$x_1, \quad x_1 + x_2, \quad x_1 + x_2 + x_3, \quad \ldots, \quad x_1 + x_2 + x_3 + \cdots + x_n, \quad \ldots$$

of partial sums, provided this limit exists. For ICFs the convergents fulfil the role of the partial sums and so, by analogy, we define the value of an ICF to be the limit (if it exists) of the sequence $\{C_n\}$ of its convergents.

We use the notation $\{C_n\}$ to represent the infinite sequence $C_1, C_2, C_3, \ldots$.

Definition 2.2 Value of an Infinite Continued Fraction

The *value* of the infinite continued fraction $[a_1, a_2, a_3, \ldots]$ is defined to be the limit of its sequence of convergents $\{C_n\}$.

Notice that we did not add the condition 'provided this limit exists'. This is because there is one big difference between infinite series and ICFs: whereas some infinite series converge and some do not, it transpires that for every ICF, its sequence of convergents has a limit.

We shall use the term 'converges' as an alternative to 'tends to a limit'. So the following theorem tells us that the sequence of convergents of every ICF converges!

Theorem 2.1 Sequences of convergents converge

For any ICF $[a_1, a_2, a_3, \ldots]$ its sequence of convergents $\{C_n\}$ tends to a limit.

Proof of Theorem 2.1

For any $n \geq 1$, the the first n convergents of the ICF $[a_1, a_2, a_3, \ldots]$ are $C_1, C_2, \ldots, C_{n-1}, C_n$. These are precisely the convergents of the FCF $C_n = [a_1, a_2, a_3, \ldots, a_n]$. What is more, $C_n = \dfrac{p_n}{q_n}$, where p_n and q_n are the numerators and denominators as determined by the formulae in Theorem 1.2.

Now Theorem 1.4 tells us that

$$C_1 < C_3 < \cdots < C_{2k+1} < \cdots < C_n < \cdots < C_{2k} < \cdots < C_4 < C_2,$$

where $2k + 1 < n$. As such a chain of inequalities holds for any value of n, no matter how large, we see that the odd convergents of the ICF form a sequence $C_1, C_3, C_5, \ldots$ which is strictly increasing and bounded above by C_2. A well-known theorem of analysis tells us that a strictly increasing sequence of real numbers which is bounded above is convergent, so $C_1, C_3, C_5, \ldots$ has a limit, say a.

This result usually goes by the name of The Monotone Convergence Theorem. Here, we are going to assume this result is known, along with some other results on the convergence of sequences of real numbers.

In the same way the even convergents form a strictly decreasing sequence $C_2, C_4, C_6, \ldots$ which is bounded below by C_1, and so it has a limit, say b.

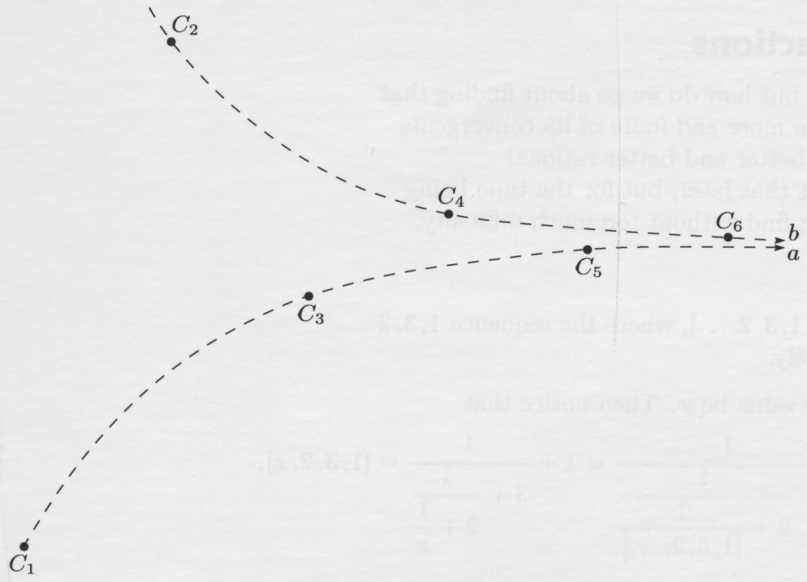


Figure 2.1

For the whole sequence $C_1, C_2, C_3, C_4, \ldots$ to converge it remains to show that $a = b$. As every even convergent is greater than every odd convergent, a and b both lie between C_{2n-1} and C_{2n}, for each positive integer n. Now

$$C_{2n} - C_{2n-1} = \frac{1}{q_{2n}q_{2n-1}}, \quad \text{by property (a) of Theorem 1.3,}$$

$$\leq \frac{1}{(2n)(2n-1)}, \quad \text{since } q_k \geq k \text{ for large } k.$$

Problem 1.10.

Therefore

$$|a - b| < C_{2n} - C_{2n-1} \leq \frac{1}{(2n)(2n-1)},$$

and, as the term on the right tends to 0 as n increases without bound, we conclude that $a = b$. ■

The above proof does more than confirm that the limit exists; it contains an estimate for the accuracy of the convergent C_n as an approximation to the actual limit. We know that the limit lies somewhere between any pair of consecutive convergents. If we couple this with the information given by Theorem 1.3 property (a), which gives a bound for how far apart successive convergents can be, we have the following immediate consequence.

Corollary to Theorem 2.1

If $\{C_n\}$ is the sequence of convergents of x then, for each $n \geq 1$:

(a) x lies between C_n and C_{n+1};

(b) $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$.

When we talk of the convergents of x we mean the convergents of the ICF of x.

$$\left|x - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}}$$

Problem 2.1

Estimate the maximum possible error in taking the convergent C_5 as an approximation to the value of $[1, 1, 1, 1, \ldots]$. What is the maximum error in C_6? The actual value of this ICF is the golden ratio. Taking the latter to be 1.61803 to five decimal places, what are the actual errors in C_5 and C_6?

2.2 Periodic continued fractions

We now know that any ICF has a value, but how do we go about finding that value? One thing we can do is determine more and more of its convergents in the knowledge that we are obtaining better and better rational approximations to its value. More about that later, but for the time being there are some ICFs whose value we can find without too much difficulty.

Example 2.1

Find the value of the ICF $[1, 3, 2, 1, 3, 2, 1, 3, 2, \ldots]$, where the sequence $1, 3, 2$ of partial quotients is repeated indefinitely.

Knowing that the ICF converges, let its value be x. Then notice that

$$1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}} = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{[1, 3, 2, \ldots]}}} = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{x}}} = [1, 3, 2, x],$$

and, as this continued fraction has value x,

$$x = [1, 3, 2, x].$$

The convergents of $[1, 3, 2, x]$ are, using the formulae for the numerators and denominators,

$$\frac{1}{1}, \frac{4}{3}, \frac{9}{7} \text{ and } \frac{9x+4}{7x+3},$$

and, as the final convergent is x itself, we have the equation

$$x = \frac{9x+4}{7x+3}; \text{ that is } 7x^2 - 6x - 4 = 0.$$

Clearly $x > a_1 = 1$ and so we want the positive root of this equation. Using the formula for the roots of a quadratic equation we obtain $x = \dfrac{3+\sqrt{37}}{7}$. ♦

Problem 2.2

Consider the ICF $[3, 1, 3, 3, 1, 3, 3, 1, 3, \ldots]$, where the sequence of partial quotients 3, 1, 3, recurs indefinitely. Determine its value x by writing

$$x = 3 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{x}}}$$

and solving the resulting quadratic equation.

Problem 2.3

Find the value of the ICF $[2, 2, 2, 2, 2, \ldots]$. From this result write down the ICF for $\sqrt{2}$ and hence find a rational number $\dfrac{a}{b}$ which lies within $\dfrac{1}{1000}$ of $\sqrt{2}$.

In Example 2.1, where we effectively wrote

$$x = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{x}}},$$

we were taking some liberties with properties of limits. To give the required argument correctly we should really be dealing with finite expressions, and subsequently taking limits in the knowledge that they do exist. That is, instead of working with the ICF whose value is x we should deal with its nth convergent, C_n. From our work on FCFs we can certainly write

$$C_n = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{C_{n-3}}}}, \quad \text{for } n \geq 4,$$

which leads (as above) to

$$C_n = \frac{9C_{n-3} + 4}{7C_{n-3} + 3}.$$

Now as n increases without bound we know that C_n converges to x, and consequently C_{n-3} converges to x. Taking limits of both sides gives

$$x = \frac{9x+4}{7x+3},$$

which is the situation we reached in Example 2.1.

The property which we have been using is the following.

Infinite continued fraction identity

$$[a_1, a_2, a_3, \ldots] = a_1 + \frac{1}{[a_2, a_3, a_4, \ldots]}.$$

This follows since we have the corresponding result for FCFs

$$[a_1, a_2, \ldots, a_n] = a_1 + \frac{1}{[a_2, a_3, \ldots, a_n]}$$

and, as n increases without bound, the FCFs on each side of this equation converge to the required ICFs.

So far, the only ICFs which we have been able to evaluate are ones in which a block of partial quotients repeat indefinitely. We have a name and notation for such ICFs.

Definition 2.3 Periodic Continued Fractions

An ICF which consists of a block of partial quotients which repeat indefinitely is said to be *purely periodic*, and we write it as $[\langle a_1, a_2, a_3, \ldots, a_n\rangle]$ enclosing the repeating block in angular brackets. We refer to the repeating block as the *cycle* of the ICF.

An ICF of the form $[b_1, b_2, \ldots, b_m, \langle a_1, a_2, \ldots, a_n\rangle]$, in which a repeating block occurs after a finite number of partial quotients, is said to be *periodic*.

For example,

$$[3, 4, 3, 4, 3, 4, \ldots] = [\langle 3, 4\rangle]$$

is a purely periodic ICF, and

$$[1, 1, 3, 4, 3, 4, 3, 4, 3, 4, \ldots] = [1, 1, \langle 3, 4\rangle]$$

is periodic. Periodic continued fractions may be written in different forms using the angle bracket notation. For example,

$$[\langle 3, 4\rangle] = [3, \langle 4, 3\rangle] = [3, 4, \langle 3, 4\rangle].$$

The evaluations that we have carried out illustrate a general strategy for finding the value of a purely periodic continued fraction. To summarize, consider $[\langle a_1, a_2, a_3, \ldots, a_n\rangle]$ with value x. The key step is the observation that x can be written as the finite (but not simple) continued fraction

$$x = [a_1, a_2, a_3, \ldots, a_n, x].$$

Adopting the usual notation for the convergents, the final two convergents of $[a_1, a_2, a_3, \ldots, a_n]$ are $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$, and can be determined by the methods of the previous section. The final convergent of $[a_1, a_2, a_3, \ldots, a_n, x]$ is then $\frac{xp_n + p_{n-1}}{xq_n + q_{n-1}}$, and equating this to x yields the quadratic equation

$$q_n x^2 + (q_{n-1} - p_n)x - p_{n-1} = 0.$$

As the first partial quotient a_1 reappears infinitely often as a partial quotient it has to satisfy $a_1 \geq 1$, and so the root we seek is certainly positive. In fact this quadratic has one positive and one negative root since the product of its roots is equal to $-\frac{p_{n-1}}{q_n}$ which is negative since p_{n-1} and q_n are each positive.

If $ax^2 + bx + c = 0$ has roots α and β then $\alpha\beta = \frac{c}{a}$ and $\alpha + \beta = -\frac{b}{a}$.

It is only a short step from the evaluation of purely periodic continued fractions to the evaluation of any periodic continued fraction, as illustrated by the following example.

Example 2.2

Evaluate the periodic continued fraction $[-2, 1, \langle 3, 4\rangle]$.

Let $y = [-2, 1, \langle 3, 4\rangle]$ and $x = [\langle 3, 4\rangle]$, so that $y = [-2, 1, x]$.

We evaluate the purely periodic x by noting that $x = [3, 4, x]$, the latter ICF having convergents $\frac{3}{1}$, $\frac{13}{4}$ and $\frac{13x+3}{4x+1}$. Equating the final convergent to x gives the quadratic

$$4x^2 - 12x - 3 = 0,$$

and this has positive root $x = \dfrac{3+2\sqrt{3}}{2}$.

Turn now to $y = [-2, 1, x]$. It has convergents $\frac{-2}{1}$, $\frac{-1}{1}$ and $\frac{-x-2}{x+1}$, and so

$$y = \frac{-x-2}{x+1} = \frac{-\dfrac{3+2\sqrt{3}}{2} - 2}{\dfrac{3+2\sqrt{3}}{2} + 1} = \frac{-7-2\sqrt{3}}{5+2\sqrt{3}}$$

$$= \frac{-7-2\sqrt{3}}{5+2\sqrt{3}} \times \frac{5-2\sqrt{3}}{5-2\sqrt{3}}$$

$$= \frac{-23+4\sqrt{3}}{13}.$$

♦

Note carefully the final step in the calculation in this example. In general the expression $\dfrac{a+b\sqrt{n}}{c+d\sqrt{n}}$ may be simplified by multiplying top and bottom by $c - d\sqrt{n}$ and appealing to the identity

$$(c+d\sqrt{n})(c-d\sqrt{n}) = c^2 - nd^2.$$

The net result is that all square-roots will be removed from the resulting denominator.

The method illustrated in Example 2.2 can be used to evaluate any periodic ICF.

Problem 2.4

Evaluate the following ICFs.

(a) $[2, \langle 1, 1, 1, 4\rangle]$ (b) $[2, 2, \langle 3\rangle]$

3 THE CONTINUED FRACTION ALGORITHM

3.1 The continued fraction of an irrational

Having seen that every finite simple continued fraction has a rational value and, conversely, that every rational number can be represented by a finite simple continued fraction, the following result should come as no surprise.

> ***Theorem 3.1 Irrationality of ICFs***
>
> The value of any infinite simple continued fraction is an irrational number.

You may have thought that this theorem must follow immediately, but we must exclude the possibility that a rational number could be represented by both a finite and an infinite simple continued fraction.

Proof of Theorem 3.1

Let $x = [a_1, a_2, a_3, \ldots]$ be an ICF. For each $n \geq 1$, x lies between the consecutive convergents $C_n = \dfrac{p_n}{q_n}$ and $C_{n+1} = \dfrac{p_{n+1}}{q_{n+1}}$, and so

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

See the Corollary to Theorem 2.1.

Now suppose that x is rational; say $x = \dfrac{r}{s}$, where r and s are integers with $s > 0$. Then

$$0 < \left| \frac{r}{s} - \frac{p_n}{q_n} \right| = \frac{|rq_n - sp_n|}{sq_n} < \frac{1}{q_n q_{n+1}},$$

Notice that $\dfrac{r}{s} \neq \dfrac{p_n}{q_n}$ for otherwise the continued fraction would be finite.

so that

$$0 < |rq_n - sp_n| < \frac{s}{q_{n+1}}.$$

But the denominators q_n form a strictly increasing sequence of integers and so we can choose n so that $q_{n+1} > s$. But then $|rq_n - sp_n|$ would be an integer lying strictly between 0 and 1, which is impossible. The conclusion from this is that no such integers r and s can exist; in other words, x must be irrational. ■

We turn now to the converse problem. Given an irrational number can it be expressed as a simple infinite continued fraction? In answering this question in the affirmative we shall discover more: the simple continued fraction of an irrational number is unique. Our approach will be to demonstrate a method, called the *Continued Fraction Algorithm*, for determining an ICF for any given irrational number. Of course we cannot determine *all* its partial quotients unless there is some recurring pattern such as occurs with periodic ICFs. What the algorithm does is enable us to determine as many partial quotients as we please. The situation is analogous to the problem of finding the decimal expansion of π; the full expansion of π cannot be given, but π can be determined to any required number of decimal places.

Given an irrational number x_1 we seek integers $a_1, a_2, \ldots$ so that $x_1 = [a_1, a_2, \ldots]$.

Knowing that x_1 lies strictly between any successive pair of its convergents, it has to lie between $C_1 = a_1$ and $C_2 = a_1 + \dfrac{1}{a_2}$. In particular, as $a_2 \geq 1$, x_1 lies strictly between the integers a_1 and $a_1 + 1$. It follows that $a_1 = \text{int}(x_1)$,

Remember $\text{int}(x)$ denotes the integer part of x.

and we note, in particular, that a_1 is determined uniquely from x_1 alone. Now let us write

$$x_1 = \text{int}(x_1) + \text{frac}(x_1),$$

where $\text{frac}(x_1)$ denotes the *fractional part* of x_1. Then

$$x_1 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} = \text{int}(x_1) + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} = \text{int}(x_1) + \frac{1}{[a_2, a_3, a_4, \ldots]}$$

and so

$$[a_2, a_3, a_4, \ldots] = \frac{1}{x_1 - \text{int}(x_1)} = \frac{1}{\text{frac}(x_1)}.$$

The fractional part of an irrational number lies strictly between 0 and 1, and so its reciprocal is greater than 1.

If we write $x_2 = \dfrac{1}{\text{frac}(x_1)}$ then $x_2 = [a_2, a_3, a_4, \ldots]$ and $x_2 > 1$, as it is an irrational number greater than the positive integer a_2. Repetition of the above argument now leads to $a_2 = \text{int}(x_2)$, and so a_2 is determined uniquely from x_2 and therefore from x_1. Continuing in this way, writing $x_3 = \dfrac{1}{\text{frac}(x_2)}$ we have $x_3 = [a_3, a_4, a_5, \ldots]$ and $a_3 = \text{int}(x_3)$, etc. This is the Continued Fraction Algorithm which is summed up in the flowchart of Figure 3.1.

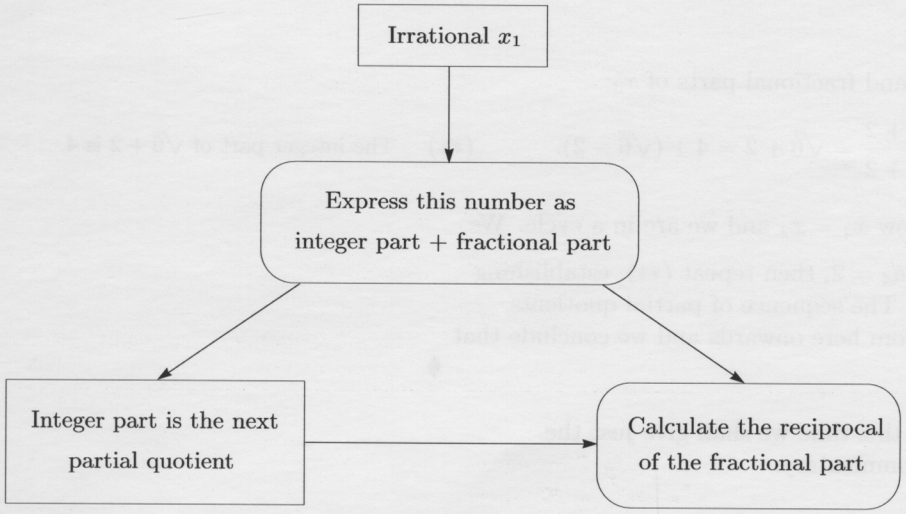


Figure 3.1 The Continued Fraction Algorithm

3.2 Using the Continued Fraction Algorithm

Before pursuing the theory let us see the algorithm in action.

Example 3.1

Determine the continued fraction of $\sqrt{6}$.

Suppose that $x_1 = \sqrt{6} = [a_1, a_2, a_3, \ldots]$ and we are to use the algorithm to determine these partial quotients. Since $2^2 < 6 < 3^2$, $\sqrt{6}$ lies between 2 and 3. Therefore $\text{int}(\sqrt{6}) = 2$ and $\sqrt{6} = 2 + (\sqrt{6} - 2)$ expresses it as integer part, 2, plus fractional part $(\sqrt{6} - 2)$. So $a_1 = 2$ and $x_2 = \dfrac{1}{\sqrt{6} - 2}$, where $x_2 = [a_2, a_3, \ldots]$.

To find a_2 we require the integer part of x_2.

$$x_2 = \frac{1}{\sqrt{6} - 2} = \frac{1}{\sqrt{6} - 2} \times \frac{\sqrt{6} + 2}{\sqrt{6} + 2} = \frac{\sqrt{6} + 2}{2}.$$

Note the trick of multiplying top and bottom by $\sqrt{6} + 2$ to remove square roots from the denominator.

Now, as $\sqrt{6}$ lies between 2 and 3, $\sqrt{6} + 2$ lies between 4 and 5, and so the integer part of $\dfrac{\sqrt{6} + 2}{2}$ is 2. Therefore

$$x_2 = 2 + \frac{\sqrt{6} - 2}{2} \qquad (*)$$

expresses x_2 as its integer part plus its fractional part. So $a_2 = 2$ and $x_3 = \dfrac{2}{\sqrt{6} - 2}$.

We continue by finding the integer and fractional parts of x_3:

$$x_3 = \frac{2}{\sqrt{6} - 2} = \frac{2}{\sqrt{6} - 2} \times \frac{\sqrt{6} + 2}{\sqrt{6} + 2} = \sqrt{6} + 2 = 4 + (\sqrt{6} - 2). \qquad (**)$$

The integer part of $\sqrt{6} + 2$ is 4.

So $a_3 = 4$ and $x_4 = \dfrac{1}{\sqrt{6} - 2}$. But now $x_4 = x_2$ and we are in a cycle. We repeat calculation $(*)$, establishing $a_4 = 2$, then repeat $(**)$, establishing $a_5 = 4$, then back to $(*)$ and so on. The sequence of partial quotients repeats the cycle 2, 4 indefinitely from here onwards and we conclude that $\sqrt{6} = [2, \langle 2, 4 \rangle]$. ♦

Our second example is similar, but this time we shall give just the calculations involved without the commentary.

Example 3.2

Determine the continued fraction of $\dfrac{7 + \sqrt{2}}{2}$.

$$x_1 = \frac{7 + \sqrt{2}}{2} = 4 + \frac{\sqrt{2} - 1}{2}; \qquad a_1 = 4, \quad x_2 = \frac{2}{\sqrt{2} - 1}$$

$$x_2 = \frac{2}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = 2\sqrt{2} + 2 = 4 + (2\sqrt{2} - 2); \quad a_2 = 4, \quad x_3 = \frac{1}{2\sqrt{2} - 2}$$

$$x_3 = \frac{1}{2\sqrt{2} - 2} \times \frac{2\sqrt{2} + 2}{2\sqrt{2} + 2} = \frac{\sqrt{2} + 1}{2} = 1 + \frac{\sqrt{2} - 1}{2}; \quad a_3 = 1, \quad x_4 = \frac{2}{\sqrt{2} - 1}$$

$x_4 = x_2$, and so $a_4 = a_2$, $a_5 = a_3$, $a_6 = a_4$, etc.

Therefore

$$\frac{7 + \sqrt{2}}{2} = [4, \langle 4, 1 \rangle]. \qquad ♦$$

Problem 3.1

Use the Continued Fraction Algorithm to determine ICFs for the following irrational numbers.

(a) $\sqrt{3}$ (b) $2+\sqrt{7}$

Having seen illustrations of the Continued Fraction Algorithm in use, we must now do a little tidying up. We have seen that the algorithm will always produce an ICF, but can we be certain that the value of this ICF is equal to the irrational number from which we started? There is also a question of uniqueness to be resolved; is the ICF of an irrational number unique?

> ***Theorem 3.2 Uniqueness of the ICF***
>
> Given any irrational number x, the ICF determined from x by the Continued Fraction Algorithm converges to x, and no other ICF converges to x.

Proof of Theorem 3.2

From the irrational number x the Continued Fraction Algorithm produces a sequence of (non-simple) finite continued fractions as follows:

$$x = x_1 = [a_1, x_2] = [a_1, a_2, x_3] = [a_1, a_2, a_3, x_4] = \cdots = [a_1, a_2, a_3, \ldots, a_n, x_{n+1}],$$

where, for all $n \geq 1$, $a_n = \text{int}(x_n)$ and the step from one stage to the next is accomplished by

$$x_n = a_n + \frac{1}{x_{n+1}}.$$

To show that the ICF which is being built up in this way has limit x we must show that the sequence of finite continued fractions

$$[a_1], \quad [a_1, a_2], \quad \ldots, \quad [a_1, a_2, \ldots, a_n], \quad \ldots$$

converges to x.

Using the standard notation for the numerators and denominators, $[a_1, a_2, \ldots, a_k] = \dfrac{p_k}{q_k}$, we have for $n \geq 2$

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

By the Corollary to Theorem 2.1.

As the denominator sequence $\{q_n\}$ is, for $n \geq 2$, a strictly increasing sequence of positive integers, $\dfrac{1}{q_n q_{n+1}}$ tends to 0, confirming that $\dfrac{p_n}{q_n}$ can be made as close as we please to x by choosing n sufficiently large. Hence the ICF determined from x by the algorithm does indeed converge to x.

It remains to show that the ICF of an irrational number is unique. To that end suppose that the ICFs $[a_1, a_2, a_3, \ldots]$ and $[b_1, b_2, b_3, \ldots]$ have the same value. We shall use the Second Principle of Mathematical Induction to show that $a_n = b_n$ for all $n \geq 1$, and so the two ICFs are identical.

First we observe that if $[a_1, a_2, a_3, \ldots] = [b_1, b_2, b_3, \ldots]$ then $a_1 = b_1$ since both are equal to the integer part of the common value. This gives the basis for the induction.

For the induction step suppose that, for some $k \geq 1$,

$$a_1 = b_1, \ a_2 = b_2, \ \ldots, \ a_k = b_k.$$

Our goal is to deduce that $a_{k+1} = b_{k+1}$. Now,

$$[a_1, a_2, a_3, \ldots] = [a_1, a_2, \ldots, a_k, [a_{k+1}, a_{k+2}, \ldots]]$$

and similarly

$$[b_1, b_2, b_3, \ldots] = [b_1, b_2, \ldots, b_k, [b_{k+1}, b_{k+2}, \ldots]].$$

As these two have the same value, and have the same first k partial quotients, it follows that

$$[a_{k+1}, a_{k+2}, \ldots] = [b_{k+1}, b_{k+2}, \ldots].$$

But now $a_{k+1} = b_{k+1}$ as each is equal to the integer part of the value of this ICF, and the induction step is complete. Hence the two ICFs having the same value are the same. ■

This proof is formalizing the repeated use of the infinite continued fraction identity. From $[a_1, a_2, a_3, \ldots] = [b_1, b_2, b_3, \ldots]$ we deduce that $a_1 = b_1$ and $[a_2, a_3, a_4, \ldots] = [b_2, b_3, b_4, \ldots]$. But then $a_2 = b_2$ and $[a_3, a_4, \ldots] = [b_3, b_4, \ldots]$ etc.

3.3 The continued fraction of $\sqrt{n}$

Let us take stock of where we have reached with simple continued fractions. For FCFs, the value of any FCF is rational and conversely every rational can be represented as an FCF. However, the representation is not unique; indeed, each rational number can be represented as an FCF in exactly two ways.

Turning to ICFs, the value of any ICF is irrational, and conversely every irrational number can be represented as a ICF. But this time the representation is unique; every irrational number has a unique ICF.

The only ICFs which we have been able to evaluate so far have been periodic ones, and these have always turned out to involve square roots in some way. We are not going to digress to prove it here, but it turns out that the values of periodic continued fractions are precisely the numbers of the form $r + s\sqrt{n}$, where r and s are rational and n is a non-square positive integer. That is, each periodic ICF has value equal to one of these numbers and, conversely, each such number has a periodic ICF. For the present, let us investigate further the ICF of certain square roots.

These numbers are called the *quadratic irrationals* and arise as roots of quadratic equations $ax^2 + bx + c = 0$, where a, b and c are integers.

Example 3.3

For each positive integer n determine the ICF for $\sqrt{n^2+1}$.

The integer part of $\sqrt{n^2+1}$ is n, and so we write

$$\sqrt{n^2+1} = n + \left(\sqrt{n^2+1} - n\right).$$

This gives $a_1 = n$. We look next at the reciprocal of $\sqrt{n^2+1} - n$, which is simplified by multiplying top and bottom by $\sqrt{n^2+1} + n$.

$$\frac{1}{\sqrt{n^2+1} - n} = \sqrt{n^2+1} + n = 2n + \left(\sqrt{n^2+1} - n\right).$$

So $a_2 = 2n$, and the same fractional part $\sqrt{n^2+1} - n$ has arisen. This means that we are immediately in a cycle and so $\sqrt{n^2+1} = [n, \langle 2n \rangle]$.

For example, for $n = 1$, 2, 3 and 4 we obtain, respectively, the following.

$$\sqrt{2} = [1, \langle 2 \rangle]$$
$$\sqrt{5} = [2, \langle 4 \rangle]$$
$$\sqrt{10} = [3, \langle 6 \rangle]$$
$$\sqrt{17} = [4, \langle 8 \rangle]$$

♦

Problem 3.2

Determine the continued fraction of $\sqrt{n^2+2}$ for any integer $n \geq 1$.

Hence find ICFs for, $\sqrt{3}$, $\sqrt{6}$, $\sqrt{11}$ and $\sqrt{18}$.

By now you should be feeling confident that you could use the Continued Fraction Algorithm to determine the ICF of $\sqrt{n}$, for any positive integer n. In the table below we collate some of the ICFs of $\sqrt{n}$ which we have seen and give several more.

Table 3.1 Simple continued fractions for $\sqrt{n}$

$\sqrt{2} = [1, \langle 2 \rangle]$	$\sqrt{13} = [3, \langle 1, 1, 1, 1, 6 \rangle]$	$\sqrt{23} = [4, \langle 1, 3, 1, 8 \rangle]$
$\sqrt{3} = [1, \langle 1, 2 \rangle]$	$\sqrt{14} = [3, \langle 1, 2, 1, 6 \rangle]$	$\sqrt{24} = [4, \langle 1, 8 \rangle]$
$\sqrt{5} = [2, \langle 4 \rangle]$	$\sqrt{15} = [3, \langle 1, 6 \rangle]$	$\sqrt{26} = [5, \langle 10 \rangle]$
$\sqrt{6} = [2, \langle 2, 4 \rangle]$	$\sqrt{17} = [4, \langle 8 \rangle]$	$\sqrt{27} = [5, \langle 5, 10 \rangle]$
$\sqrt{7} = [2, \langle 1, 1, 1, 4 \rangle]$	$\sqrt{18} = [4, \langle 4, 8 \rangle]$	$\sqrt{28} = [5, \langle 3, 2, 3, 10 \rangle]$
$\sqrt{8} = [2, \langle 1, 4 \rangle]$	$\sqrt{19} = [4, \langle 2, 1, 3, 1, 2, 8 \rangle]$	$\sqrt{29} = [5, \langle 2, 1, 1, 2, 10 \rangle]$
$\sqrt{10} = [3, \langle 6 \rangle]$	$\sqrt{20} = [4, \langle 2, 8 \rangle]$	$\sqrt{30} = [5, \langle 2, 10 \rangle]$
$\sqrt{11} = [3, \langle 3, 6 \rangle]$	$\sqrt{21} = [4, \langle 1, 1, 2, 1, 1, 8 \rangle]$	$\sqrt{31} = [5, \langle 1, 1, 3, 5, 3, 1, 1, 10 \rangle]$
$\sqrt{12} = [3, \langle 2, 6 \rangle]$	$\sqrt{22} = [4, \langle 1, 2, 4, 2, 1, 8 \rangle]$	$\sqrt{32} = [5, \langle 1, 1, 1, 10 \rangle]$

$\vdots$

$\sqrt{46} = [6, \langle 1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12 \rangle]$
$\sqrt{61} = [7, \langle 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14 \rangle]$
$\sqrt{94} = [9, \langle 1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18 \rangle]$
$\sqrt{97} = [9, \langle 1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18 \rangle]$
$\sqrt{151} = [12, \langle 3, 2, 7, 1, 3, 4, 1, 1, 1, 11, 1, 1, 1, 4, 3, 1, 7, 2, 3, 24 \rangle]$
$\sqrt{166} = [12, \langle 1, 7, 1, 1, 1, 2, 4, 1, 3, 2, 12, 2, 3, 1, 4, 2, 1, 1, 1, 7, 1, 24 \rangle]$

Problem 3.3

Examine Table 3.1.

(a) What common features do you observe in the continued fractions?

(b) Can you draw any conclusions concerning the continued fraction of $\sqrt{n} + \text{int}(\sqrt{n})$?

(c) Conjecture the general form of the continued fraction of $\sqrt{n}$, for n a positive non-square integer.

From our solution to Problem 3.3 we are led to believe that, for any non-square positive integer n, the ICF of $\sqrt{n}$ takes the form

$$\sqrt{n} = [a_1, \langle a_2, a_3, \ldots, a_3, a_2, 2a_1 \rangle],$$

where a_1 is the integer part of $\sqrt{n}$. Here the periodic part may consist of:

- just $\langle 2a_1 \rangle$ as in, for example, $\sqrt{2} = [1, \langle 2 \rangle]$ and $\sqrt{17} = [4, \langle 8 \rangle]$;
- an odd number of partial quotients as in $\sqrt{13} = [3, \langle 1, 1, 1, 1, 6 \rangle]$;
- an even number of partial quotients as in $\sqrt{7} = [2, \langle 1, 1, 1, 4 \rangle]$.

It transpires that the general form is, indeed, as claimed. However proof of this would take us deep into the study of more properties of continued fractions, which we do not have space for in this course.

4 RATIONAL APPROXIMATIONS

4.1 Approximating irrationals by rationals

We have seen that an irrational number has a unique ICF and that the convergents provide a sequence of better and better rational approximations to the irrational number. We even have a way of estimating how good these rational approximations are. In this section we shall go further and show that continued fractions provide what are, in a certain sense, best rational approximations.

Given an irrational number x, we can always improve on any rational approximation $\frac{a}{b}$ to x. In fact there are infinitely many rational numbers lying between $\frac{a}{b}$ and x. Consequently there is no such thing as *the best* rational approximation to x. However, if we impose restrictions on the size of the denominator we can ask about best approximations. For example, of all rational numbers whose (positive) denominator does not exceed 50, the best rational approximation to π is $\frac{22}{7}$. We have to allow the denominator to climb to 57 before we find the better approximation $\frac{179}{57}$ to π.

Remember that when we write $\frac{a}{b}$ for a rational number we assume that a, b are integers with b positive and $\gcd(a,b) = 1$.

What we shall show in this section is that if $\frac{p_n}{q_n}$ is a convergent of x then there is no rational number $\frac{a}{b}$ with $0 < b < q_n$ which approximates x better than $\frac{p_n}{q_n}$, and in this sense the convergents of x are best rational approximations to it. But before becoming involved in the analysis we give an elegant geometric interpretation of the convergents to a continued fraction of an irrational number which was first given by Klein in 1895. It illustrates precisely the sense in which convergents are best rational approximations.

Let $\alpha > 0$ be irrational and consider the part of the line $y = \alpha x$ in the first quadrant of the Cartesian plane as illustrated in Figure 4.1. This line cannot pass through any of the lattice points (i.e points whose coordinates are both integers) except $(0,0)$, for if it did α would be rational. Imagine that pins are set upright at each of the lattice points and a fine string is placed along the line with one end fixed to the line at an infinitely remote point, and the other end at the origin. Keeping the string tight we move the end at the origin towards the right. It will catch on a series of pins below the line. In fact, the pins on which it catches are those at lattice points with coordinates (q_1, p_1), (q_3, p_3), $(q_5, p_5), \ldots$ corresponding to the odd convergents

$$\frac{p_1}{q_1}, \frac{p_3}{q_3}, \frac{p_5}{q_5}, \ldots$$

of α. Similarly, if we move the string from the origin to the left it catches on pegs at (q_2, p_2), (q_4, p_4), (q_6, p_6), ..., corresponding to the even convergents

$$\frac{p_2}{q_2}, \frac{p_4}{q_4}, \frac{p_6}{q_6}, \ldots$$

of α.

If $y = \alpha x$ passes through (m, n) then $\alpha = \frac{n}{m}$.

Recall that

$$C_1 < C_3 < C_5 < \cdots < \alpha < \cdots < C_4 < C_2,$$

and notice that $C_i = \frac{p_i}{q_i}$ give the gradients of the lines joining the origin $(0,0)$ to the points (q_i, p_i).

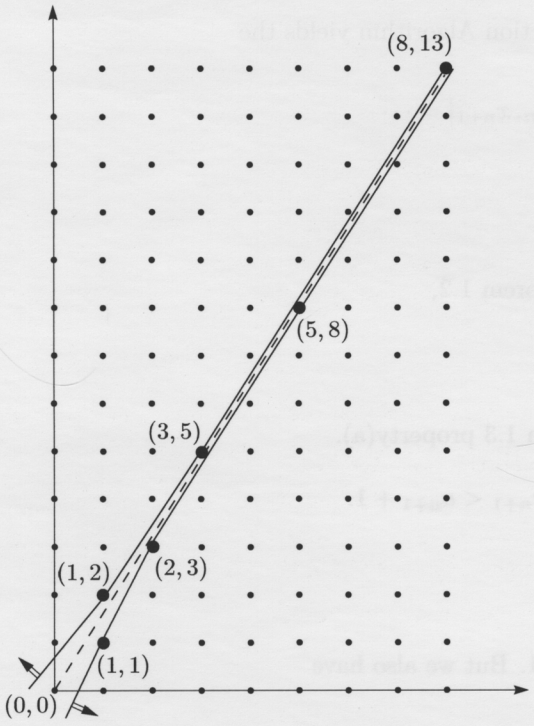


Figure 4.1 Convergents as best approximations to the golden ratio

Figure 4.1 illustrates all this for the case $\alpha = \dfrac{1+\sqrt{5}}{2}$, the golden ratio, whose convergents begin $\dfrac{1}{1}, \dfrac{2}{1}, \dfrac{3}{2}, \dfrac{5}{3}, \dfrac{8}{5}, \dfrac{13}{8}, \ldots$. The dashed line is the portion of $y = \alpha x$ in the first quadrant. Moving to the right it catches on pins with coordinates $(1,1), (2,3), (5,8), \ldots$, whilst moving left it catches on the pins at $(1,2), (3,5), (8,13)$, and so on.

Let us return to the question of best rational approximations. First we must prove, formally, a result which we have seen illustrated in the many sequences of convergents that we have calculated; each convergent is a better approximation to the irrational number than is its predecessor. The theorem which follows includes this result, but also gives some bounds for the accuracy of the convergents.

Theorem 4.1 Accuracy of convergents

If $\dfrac{p_1}{q_1}, \dfrac{p_2}{q_2}, \dfrac{p_3}{q_3}, \ldots$ is the sequence of convergents of x then, for all $k \geq 1$,

$$\left| x - \frac{p_{k+1}}{q_{k+1}} \right| \leq \frac{1}{q_{k+1}q_{k+2}} \leq \frac{1}{2q_kq_{k+1}} < \left| x - \frac{p_k}{q_k} \right|.$$

Proof of Theorem 4.1

Let x have ICF $[a_1, a_2, a_3, \ldots]$. The Continued Fraction Algorithm yields the system of equations

$$x = [a_1, x_2] = [a_1, a_2, x_3] = \cdots = [a_1, a_2, \ldots, a_n, x_{n+1}] = \cdots$$

and

$$\begin{aligned}\left|x - \frac{p_n}{q_n}\right| &= \left|[a_1, a_2, \ldots, a_n, x_{n+1}] - \frac{p_n}{q_n}\right| \\ &= \left|\frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n}\right|, \quad \text{by Theorem 1.2,} \\ &= \left|\frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(x_{n+1}q_n + q_{n-1})}\right| \\ &= \frac{1}{q_n(x_{n+1}q_n + q_{n-1})}, \quad \text{by Theorem 1.3 property(a).}\end{aligned}$$

Now $x_{n+1} = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots]$ and so $a_{n+1} < x_{n+1} < a_{n+1} + 1$.

Therefore

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_nq_{n+1}},$$

which gives the left-hand inequality when $n = k + 1$. But we also have

$$\begin{aligned}\left|x - \frac{p_n}{q_n}\right| &> \frac{1}{q_n((a_{n+1}+1)q_n + q_{n-1})} \\ &= \frac{1}{q_n(a_{n+1}q_n + q_{n-1}) + q_n^2} = \frac{1}{q_n(q_{n+1} + q_n)} \\ &> \frac{1}{q_n(q_{n+1} + q_{n+1})} = \frac{1}{2q_nq_{n+1}},\end{aligned}$$

Remember that $\{q_n\}$, for $n \geq 2$, is a strictly increasing sequence.

which gives the right-hand inequality by taking $n = k$.

For the middle inequality note that

$$q_{k+2} = a_{k+2}q_{k+1} + q_k \geq q_k + q_k = 2q_k$$

and so

$$\frac{1}{q_{k+1}q_{k+2}} \leq \frac{1}{2q_{k+1}q_k}.$$

■

The left-hand inequality in Theorem 4.1 gives an indication of how close each convergent is to the true value whilst the right-hand inequality gives a bound which limits the accuracy. We can collate the two facts for each convergent as follows.

Corollary to Theorem 4.1

For all $k \geq 1$ the convergent $\dfrac{p_k}{q_k}$ of x satisfies

$$\frac{1}{2q_kq_{k+1}} < \left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_kq_{k+1}}.$$

Problem 4.1

The ICF of $\sqrt{5}$ is $[2, \langle 4 \rangle]$. Estimate the accuracy of the third convergent as an approximation to $\sqrt{5}$.

4.2 Best approximations

The discovery that the ICFs of $\sqrt{n}$ can be described exactly in terms of finite recurring cycles is an interesting fact, because the decimal representations of these numbers (when n is non-square) have no discernible pattern. Of course ICFs may have other patterns besides recurring blocks which permit them to be described exactly. Amazingly, one of the best-known of all irrational numbers, e, the base of the natural logarithms, has such a pattern to its ICF. Euler discovered in 1737 that

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \ldots],$$

where the even positive integers 2, 4, 6, ... occur in order as every third partial quotient, the other partial quotients all being 1 with the exception of $a_1 = 2$.

Example 4.1

Determine the first convergent of e which is accurate to within 0.0001 of the true value of e.

The convergents of e are

$$\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \frac{1264}{465}, \ldots.$$

Using the result of the Corollary we know that the convergent $\frac{11}{4}$, lies within $\frac{1}{4 \times 7}$ of e, $\frac{19}{7}$ lies within $\frac{1}{7 \times 32}$ of e, and so on. The first convergent guaranteed to be accurate to 0.0001 of e is $\frac{193}{71}$ which lies within $\frac{1}{71 \times 465} = 0.0000303\ldots$ of e. The previous convergent $\frac{106}{39}$ does not have the required accuracy because, using the left-hand inequality in the Corollary, the error in $\frac{106}{39}$ is at least

$$\frac{1}{2 \times 39 \times 71} = 0.00018\ldots.$$

♦

The values to six decimal places are as follows.

$$e = 2.718282$$
$$\frac{193}{71} = 2.718310$$
$$\frac{106}{39} = 2.717949$$

Problem 4.2

The ICF of $\sqrt{2}$ is $[1, \langle 2 \rangle]$. Determine the first convergent which is accurate to within 0.0001 of the true value of $\sqrt{2}$.

We now appreciate that the convergent $\frac{193}{71}$ is a very good approximation to e, but as yet we have made little progress towards establishing the sense in which it is 'best'. We shall put this right immediately by showing that of all rational numbers with denominator not exceeding 71, the convergent $\frac{193}{71}$ is the best approximation. The general result from which we can draw this as a conclusion follows quickly from the slightly stronger result of Theorem 4.2 below.

For completeness we have given proofs of all the remaining results in this section but you should regard these proofs as optional reading.

Theorem 4.2

Let $\frac{p_n}{q_n}$ be the nth convergent of the irrational number x, and let $\frac{a}{b}$ be any rational number with $0 < b < q_{n+1}$. Then, for $n > 1$,

$$|q_n x - p_n| \leq |bx - a|,$$

the equality holding only if $a = p_n$ and $b = q_n$.

Remember $\frac{a}{b}$ is in lowest terms.

Proof of Theorem 4.2

Let $\frac{a}{b}$ be a rational number (in its lowest terms) with $b < q_{n+1}$, and suppose it is not the case that $a = p_n$ and $b = q_n$, when the required equality certainly holds.

Consider the system of equations (with r, s real for the moment):

$$a = rp_n + sp_{n+1};$$
$$b = rq_n + sq_{n+1}.$$

Multiplying the first equation by q_n and the second by p_n and subtracting leads to

$$aq_n - bp_n = s(p_{n+1}q_n - p_nq_{n+1})$$

which, after applying Theorem 1.3 property (a) to the bracketed term on the right-hand side, gives

$$s = (-1)^{n+1}(aq_n - bp_n).$$

Similarly

$$r = (-1)^{n+1}(bp_{n+1} - aq_{n+1}).$$

Note that r and s are therefore integers, and neither of them is 0 because the assumption that $r = 0$ leads to $aq_{n+1} = bp_{n+1}$, whereupon Euclid's Lemma forces q_{n+1} to divide b, since $\gcd(p_{n+1}, q_{n+1}) = 1$, and this contradicts $0 < b < q_{n+1}$. Also if $s = 0$ we have $\frac{a}{b} = \frac{p_n}{q_n}$, which is the case of equality which has been excluded.

Now since $0 < b = rq_n + sq_{n+1} < q_{n+1}$, the non-zero integers r and s must have opposite sign. Moreover $x - \frac{p_n}{q_n}$ and $x - \frac{p_{n+1}}{q_{n+1}}$ have opposite signs because the convergents are alternatively greater than and then less than the irrational limit x. It follows that $r(q_nx - p_n)$ and $s(q_{n+1}x - p_{n+1})$ have the same sign. (We need this fact so that we can replace the absolute value of their sum by the sum of their absolute values.)

$$\begin{aligned} |bx - a| &= |(rq_n + sq_{n+1})x - (rp_n + sp_{n+1})| \\ &= |r(q_nx - p_n) + s(q_{n+1}x - p_{n+1})| \\ &= |r||q_nx - p_n| + |s||q_{n+1}x - p_{n+1}| \\ &> |r||q_nx - p_n| \geq |q_nx - p_n| \end{aligned}$$

as claimed. ■

As a consequence of this result we can now spell out the sense in which the convergents are best approximations to the irrational number.

Corollary Convergents are best approximations

Let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be the convergents of the irrational number x. Then, for any rational number $\frac{a}{b}$ with $1 \leq b \leq q_n$

$$\left|x - \frac{p_n}{q_n}\right| \leq \left|x - \frac{a}{b}\right|,$$

the equality holding only if $a = p_n$ and $b = q_n$.

Remember $\frac{a}{b}$ is in lowest terms.

This means for example, that as $\frac{1264}{465}$ is a convergent of e there is no better rational approximation $\frac{a}{b}$ to e with $0 < b \leq 465$.

Proof of the Corollary

Assume, to the contrary, that there is a rational number $\frac{a}{b}$ with $1 \leq b \leq q_n$ and

$$\left| x - \frac{a}{b} \right| < \left| x - \frac{p_n}{q_n} \right|.$$

Then

$$|q_n x - p_n| = q_n \left| x - \frac{p_n}{q_n} \right| > q_n \left| x - \frac{a}{b} \right| \geq b \left| x - \frac{a}{b} \right| = |bx - a|,$$

contradicting the result of the theorem. ■

4.3 Rational approximations to π

Having seen that there are patterns for the ICFs of all quadratic irrationals and also for the irrational e, we would be lacking curiosity if we did not enquire about the ICF of the most famous of all irrationals, π.
Disappointingly, it transpires that there is no known pattern to the sequence of partial quotients of π. We can see how the ICF begins by working with some good known approximation to π. For example, take $\pi = 3.141592654$. With the aid of a calculator, the continued fraction algorithm yields the following.

$$\pi = 3 + (\pi - 3) = 3 + 0.141592654; \qquad a_1 = 3$$

$$\frac{1}{0.141592654} = 7.062513305; \qquad a_2 = 7$$

$$\frac{1}{0.0625133059} = 15.99659454; \qquad a_3 = 15$$

$$\frac{1}{0.99659454} = 1.003417099; \qquad a_4 = 1$$

$$\frac{1}{0.003417099} = 292.6458622; \qquad a_5 = 292$$

Calculations like these are very sensitive to rounding so your calculator may well produce different results from ours.

This, of course, is only giving a crude idea of how the ICF of π begins. Here we are not calculating the ICF of π itself but rather the continued fraction of the rational approximation to nine decimal places with which we began, and as we are not calculating the reciprocals exactly, rounding errors are creeping in. For example, if we were to use a calculator which works only to eight decimal places the ensuing value of a_5 would turn out to be significantly different from 292. But the partial quotients we have obtained thus far are correct for π.

$$\pi = [3, 7, 15, 1, 292, \ldots].$$

The resulting sequence of convergents begins

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103\,993}{33\,102}, \ldots.$$

The search for good rational approximations to π has a long history. With the aid of today's computers the value of π is now known correct to millions of decimal places. The first known rational approximations can be traced back to Archimedes (287–212 BC). Archimedes utilized the fact that the circumference of the circle lies between the total edge length of any inscribed polygon and that of any circumscribed polygon. By using a polygon of 96 sides he discovered lower and upper bounds given by

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

The value $\frac{22}{7}$, known as the *Archimedean value of* π, is commonly used as an approximation to π, being accurate to within 0.001264. The lower bound,

$\frac{223}{71}$, is not a convergent of π but turns out to be more accurate than $\frac{22}{7}$, the error being approximately 0.0007476. The next convergent of π, $\frac{333}{106}$, must be more accurate still because we know it is the best approximation of all rational numbers with denominator not exceeding 106; the error in $\frac{333}{106}$ is approximately 0.000083.

However the next convergent, $\frac{355}{113}$, is remarkably accurate. (This value for π seems to have been first known to a Chinese mathematician Tsu Chung-Chih (430–501 AD) but was not discovered in Europe until late in the sixteenth century.) Because of the large partial quotient 292 there is a huge jump in the values of numerator and denominator following the convergent $\frac{355}{113}$, and the corollary to Theorem 4.1 gives the bound on its error:

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{113 \times 33\,102} < \frac{3}{10^7}.$$

Our theory tells us that $\frac{355}{113}$ is a better approximation than any other rational with denominator not exceeding 113. In fact this claim is not very flattering, as the following example will show.

Example 4.2

Show that $\frac{355}{113}$ is the best of all rational approximations to π with denominator not exceeding 10 000.

Suppose that $\frac{r}{s}$ is a rational number with $114 \leq s \leq 10\,000$. Then

$$\left|\frac{r}{s} - \frac{355}{113}\right| = \frac{|113r - 355s|}{113s} \geq \frac{1}{113s} \geq \frac{1}{113 \times 10\,000}.$$

But we also have

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{113 \times 33\,102}.$$

Hence $\frac{355}{113}$ lies within $\frac{1}{113 \times 33\,102}$ of π but is further than $\frac{1}{113 \times 10\,000}$ from $\frac{r}{s}$. The triangle inequality gives

$$\begin{aligned}\left|\frac{r}{s} - \pi\right| &\geq \left|\frac{r}{s} - \frac{355}{113}\right| - \left|\pi - \frac{355}{113}\right| \\ &> \frac{1}{113 \times 10\,000} - \frac{1}{113 \times 33\,102} \\ &= \frac{23\,102}{113 \times 10\,000 \times 33\,102} > \frac{1}{113 \times 33\,102}.\end{aligned}$$

Thus $\frac{r}{s}$ is not as good an approximation to π as is $\frac{355}{113}$. ♦

The triangle inequality for real numbers says

$$|a - b| \leq |a - x| + |x - b|.$$

The form used here is

$$|a - x| \geq |a - b| - |x - b|.$$

Problem 4.3

Let $x = [0, 1, 2, 3, 4, 5, 6, \ldots]$. Determine the first convergent of x which is within 10^{-5} of x. Show that of all rational numbers $\frac{r}{s}$, where the positive denominator s does not exceed 100, the most accurate approximation to x is $\frac{30}{43}$.

We have seen that the convergents to the irrational number x satisfy $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$ and, as the denominator sequence $\{q_n\}$ is increasing, $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$. A question we might ask concerns whether or not this property characterizes convergents; that is, if a rational number $\frac{a}{b}$ has the property $\left|x - \frac{a}{b}\right| \leq \frac{1}{b^2}$, is $\frac{a}{b}$ necessarily a convergent of x? It is not difficult to show that the answer is *no*. For example, $\left|\sqrt{3} - \frac{3}{2}\right| \leq \frac{1}{4}$ and yet $\frac{3}{2}$ is not a convergent of $\sqrt{3}$. Nevertheless there is an inequality of this kind which does characterize convergents.

Theorem 4.3 Which rationals are convergents?

If the rational number $\frac{a}{b}$ satisfies

$$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$$

then $\frac{a}{b}$ is a convergent of x.

Proof of Theorem 4.3

Suppose to the contrary that $\frac{a}{b}$ is a rational number (in lowest terms) satisfying the given inequality but which is not one of the convergents $\frac{p_n}{q_n}$ of x. Let r be the unique integer for which $q_r \leq b < q_{r+1}$. Then, by Theorem 4.2,

$$|q_r x - p_r| \leq |bx - a| = b\left|x - \frac{a}{b}\right| < b \times \frac{1}{2b^2} = \frac{1}{2b}.$$

Therefore

$$q_r\left|x - \frac{p_r}{q_r}\right| < \frac{1}{2b}, \text{ which gives } \left|x - \frac{p_r}{q_r}\right| < \frac{1}{2q_r b}.$$

The triangle inequality then gives

$$\left|\frac{a}{b} - \frac{p_r}{q_r}\right| \leq \left|x - \frac{p_r}{q_r}\right| + \left|x - \frac{a}{b}\right| < \frac{1}{2q_r b} + \frac{1}{2b^2}. \qquad (*)$$

On the other hand

$$\left|\frac{a}{b} - \frac{p_r}{q_r}\right| = \left|\frac{q_r a - p_r b}{q_r b}\right| \geq \frac{1}{q_r b}. \qquad (**)$$

(The final inequality comes from the fact that $q_r a - p_r b$ is an integer and is non-zero, for otherwise $\frac{a}{b}$ would be equal to the convergent $\frac{p_r}{q_r}$, which we have assumed not to be the case.) Combining $(*)$ and $(**)$ we have

$$\frac{1}{q_r b} < \frac{1}{2q_r b} + \frac{1}{2b^2},$$

and this simplifies to $q_r > b$, giving a contradiction. ■

Problem 4.4

Without calculating the ICF show that $\frac{17}{16}$ is a convergent of $\frac{2+\sqrt{11}}{5}$.

ADDITIONAL EXERCISES

Section 1

1 Express each of the following rational numbers as an FCF.

(a) $\frac{23}{54}$ (b) $\frac{117}{17}$ (c) $\frac{-37}{24}$ (d) 3.142

2 (a) If $x > 1$ and x has FCF $[a_1, a_2, a_3, \ldots a_n]$ show that $\frac{1}{x}$ has FCF $[0, a_1, a_2, a_3, \ldots a_n]$. What is the corresponding result when $0 < x < 1$?

(b) Suppose that $-1 < x < 0$ and that x has FCF $[-1, a_1, a_2, \ldots a_n]$. Show that $[a_1, a_2, \ldots a_n]$ has value $\frac{1}{1+x}$.

3 Evaluate the sequence of convergents for each of the following FCFs and, in each case, verify that Theorem 1.4 holds.

(a) $[2, 2, 4, 2, 4, 2]$ (b) $[1, 2, 3, 4, 5, 6, 7]$

4 Use continued fractions to find the general solution of each of the following linear Diophantine equations.

(a) $17x + 91y = 1$ (b) $16x + 219y = 17$

Section 2

1 Evaluate the following periodic continued fractions.

(a) $[\langle 5, 1, 3\rangle]$ (b) $[0, \langle 5, 1, 3\rangle]$ (c) $[1, 2, \langle 5, 1, 3\rangle]$

2 Euler showed that the ICF of the irrational number $\frac{e^{2/k} - 1}{e^{2/k} + 1}$ is

$$[0, k, 3k, 5k, 7k, 9k, \ldots].$$

Use this information to find values correct to four decimal places for the following.

(a) $\frac{e-1}{e+1}$ (b) $\frac{e^2-1}{e^2+1}$

Section 3

1 Determine the ICFs of the following irrational numbers.

(a) $\sqrt{11}$ (b) $\frac{1+\sqrt{13}}{2}$ (c) $\frac{4+\sqrt{21}}{5}$

2 If $0 < k \leq 2n$ and k divides $2n$, show that

$$\sqrt{n^2 + k} = \left[n, \left\langle \frac{2n}{k}, 2n \right\rangle\right].$$

Hence find ICFs for (a) $\sqrt{12}$, (b) $\sqrt{24}$.

3 Determine the ICF of $\sqrt{n^2 - n}$, for a general $n \geq 2$. Use this to write down the ICFs of $\sqrt{2}$, $\sqrt{6}$ and $\sqrt{12}$.

Section 4

1 The irrational number x has ICF $[0, 2, 4, 6, 8, 10, \ldots]$. Find the best rational approximation $\dfrac{a}{b}$ to x with:

(a) $b < 100$; (b) $b < 1000$.

2 Theorem 4.3 tells us that any rational number $\dfrac{a}{b}$ satisfying

$$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2},$$

is a convergent of x. It is not true that every convergent $\dfrac{a}{b}$ of x satisfies the inequality.

However, prove that of two consecutive convergents $\dfrac{p_n}{q_n}$ and $\dfrac{p_{n+1}}{q_{n+1}}$ of x, at least one of them does satisfy this inequality. *Hint*: Assume the contrary and use the fact that

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|x - \frac{p_{n+1}}{q_{n+1}}\right| + \left|x - \frac{p_n}{q_n}\right|.$$

3 A theorem of Hurwitz (1891) shows that for each irrational number x there are infinitely many rationals $\dfrac{a}{b}$ satisfying

$$\left|x - \frac{a}{b}\right| < \frac{1}{\sqrt{5}b^2}.$$

(a) Why must any rational number $\dfrac{a}{b}$ satisfying the above inequality be a convergent of x?

(b) Find two convergents of $\sqrt{3}$ which satisfy this inequality, and two which do not.

Challenge Problems

1 In applying the Continued Fraction Algorithm to determine the ICF of $\sqrt{n}$ we obtain the sequence of equations

$$\sqrt{n} = x_1 = [a_1, x_2] = [a_1, a_2, x_3] = [a_1, a_2, a_3, x_4] = \cdots,$$

where, for all $k \geq 1$, $a_k = \text{int}(x_k)$ and the step from one stage to the next is accomplished by

$$x_k = a_k + \frac{1}{x_{k+1}}.$$

Prove that for each $k \geq 1$,

$$x_k = \frac{r_k + \sqrt{n}}{s_k},$$

where $\{r_k\}$ and $\{s_k\}$ are sequences of integers given by

$$r_1 = 0, \quad s_1 = 1, \quad r_{k+1} = a_k s_k - r_k, \quad s_{k+1} = \frac{n - r_{k+1}^2}{s_k}.$$

Use this sequence to determine the ICF of $\sqrt{14}$.

2 Let $x = [a_1, a_2, a_3, a_4, \ldots]$. Find the ICF of $-x$.

3 Let $x = [a_1, a_2, a_3, a_4, \ldots]$. Show that provided a_2 is even, the ICF of $2x$ satisfies

$$2[a_1, a_2, a_3, a_4, \ldots] = [2a_1, \tfrac{1}{2}a_2, 2[a_3, a_4, a_5, \ldots]], \qquad (*)$$

and deduce that provided all the even numbered partial quotients are even,

$$2[a_1, a_2, a_3, a_4, \ldots] = [2a_1, \tfrac{1}{2}a_2, 2a_3, \tfrac{1}{2}a_4, 2a_5, \tfrac{1}{2}a_6, \ldots].$$

Find a formula corresponding to $(*)$ for the case of a_2 being odd.

4 Let $\boldsymbol{A}_n = \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$ and $\boldsymbol{M}_n = \begin{pmatrix} q_n & p_n \\ q_{n+1} & p_{n+1} \end{pmatrix}$, where $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots$ are the convergents of $[a_1, a_2, a_3, a_4, \ldots]$. Show the following.

(a) $\boldsymbol{M}_n = \boldsymbol{A}_{n+1}\boldsymbol{M}_{n-1}$, for $n > 1$

(b) $\boldsymbol{M}_n = \boldsymbol{A}_{n+1}\boldsymbol{A}_n\boldsymbol{A}_{n-1}\ldots\boldsymbol{A}_2\boldsymbol{A}_1$

(c) By considering the determinant of $\boldsymbol{M}_n$, deduce the formula of Theorem 1.3 property(a):

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n, \quad \text{for } n \geq 2.$$

(d) By considering $\boldsymbol{M}_n$ and its transpose, deduce the formulae

$$\frac{p_n}{p_{n-1}} = [a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1]$$

and

$$\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, a_{n-2}, \ldots, a_3, a_2].$$

The transpose of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

SOLUTIONS TO THE PROBLEMS

Solution 1.1

(a) $$[2,3,4] = 2 + \cfrac{1}{3 + \cfrac{1}{4}} = 2 + \frac{4}{13} = \frac{30}{13}$$

(b) $$[0,2,2,2] = 0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 0 + \frac{1}{2 + \frac{2}{5}} = 0 + \frac{5}{12} = \frac{5}{12}$$

Solution 1.2

(a) $11 = 1 \times 7 + 4$

$7 = 1 \times 4 + 3$

$4 = 1 \times 3 + 1$

$3 = 3 \times 1$

So $\frac{11}{7} = [1,1,1,3]$.

(b) $43 = 2 \times 19 + 5$

$19 = 3 \times 5 + 4$

$5 = 1 \times 4 + 1$

$4 = 4 \times 1$

So $\frac{43}{19} = [2,3,1,4]$.

Solution 1.3

(a) $65 = 2 \times 23 + 19$
$23 = 1 \times 19 + 4$
$19 = 4 \times 4 + 3$
$4 = 1 \times 3 + 1$
$3 = 3 \times 1$
So $\dfrac{65}{23} = [2, 1, 4, 1, 3]$.

(b) $23 = 0 \times 53 + 23$
$53 = 2 \times 23 + 7$
$23 = 3 \times 7 + 2$
$7 = 3 \times 2 + 1$
$2 = 2 \times 1$
So $\dfrac{23}{53} = [0, 2, 3, 3, 2]$.

(c) $-94 = -8 \times 13 + 10$
$13 = 1 \times 10 + 3$
$10 = 3 \times 3 + 1$
$3 = 3 \times 1$
So $\dfrac{-94}{13} = [-8, 1, 3, 3]$.

Note that the integer part of $-\dfrac{94}{13}$ is -8 (not -7).

Solution 1.4

(a) $[1, 2, 3] = [1, 2, 2, 1]$
(b) $[-4, 3, 1, 6, 1] = [-4, 3, 1, 7]$
(c) $[0] = [-1, 1]$

Solution 1.5

(a) The numerators of $[2, 3, 4, 1, 2]$ are

$$p_1 = 2; \quad p_2 = 2 \times 3 + 1 = 7; \quad p_3 = 4 \times 7 + 2 = 30;$$
$$p_4 = 1 \times 30 + 7 = 37; \quad p_5 = 2 \times 37 + 30 = 104.$$

The denominators are

$$q_1 = 1; \quad q_2 = 3; \quad q_3 = 4 \times 3 + 1 = 13;$$
$$q_4 = 1 \times 13 + 3 = 16; \quad q_5 = 2 \times 16 + 13 = 45.$$

The convergents are therefore

$$\frac{2}{1}, \frac{7}{3}, \frac{30}{13}, \frac{37}{16} \text{ and } \frac{104}{45}.$$

(b) The numerators of $[-2, 3, 1, 6]$ are

$$p_1 = -2; \quad p_2 = (-2) \times 3 + 1 = -5;$$
$$p_3 = 1 \times (-5) + (-2) = -7; \quad p_4 = 6 \times (-7) + (-5) = -47.$$

The denominators are

$$q_1 = 1; \quad q_2 = 3; \quad q_3 = 1 \times 3 + 1 = 4; \quad q_4 = 6 \times 4 + 3 = 27.$$

The convergents are therefore

$$\frac{-2}{1}, \frac{-5}{3}, \frac{-7}{4} \text{ and } \frac{-47}{27}.$$

Solution 1.6

(a)

k	1	2	3	4	5
p_k	-4	-3	-19	-22	-151
a_k	-4	1	5	1	6
q_k	1	1	6	7	48

The convergents are

$$\frac{-4}{1}, \frac{-3}{1}, \frac{-19}{6}, \frac{-22}{7} \text{ and } \frac{-151}{48}.$$

(b)

k	1	2	3	4	5	6	7
p_k	1	2	5	7	19	26	71
a_k	1	1	2	1	2	1	2
q_k	1	1	3	4	11	15	41

The convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15} \text{ and } \frac{71}{41}.$$

Solution 1.7

(a) The convergents of $[2, 3, 4, 1, 2]$ were found to be

$$\frac{2}{1}, \frac{7}{3}, \frac{30}{13}, \frac{37}{16} \text{ and } \frac{104}{45}.$$

The FCF $[2, 3, 4, 1, 2, 2]$ has the same first five convergents plus a sixth, which is its value. The latter is determined from the previous convergents as

$$\frac{2 \times 104 + 37}{2 \times 45 + 16} = \frac{245}{106}.$$

(b) Similarly $[2, 3, 4, 1, 2, 100]$ has the same first five convergents as $[2, 3, 4, 1, 2]$ and has final convergent

$$\frac{100 \times 104 + 37}{100 \times 45 + 16} = \frac{10437}{4516}.$$

The addition of a further, even larger, partial quotient has not changed the value of the FCF by much. In fact $\frac{104}{45} \doteq 2.3111111$, $\frac{245}{106} \doteq 2.3113208$ and $\frac{10437}{4516} \doteq 2.3111160$.

Solution 1.8

From the Euclidean algorithm

$$48 = 2 \times 19 + 10$$
$$19 = 1 \times 10 + 9$$
$$10 = 1 \times 9 + 1$$
$$9 = 9 \times 1$$

and so $\frac{48}{19}$ has FCF $[2, 1, 1, 9]$. Its convergents are

$$\frac{2}{1}, \frac{3}{1}, \frac{5}{2} \text{ and } \frac{48}{19}.$$

From the final two convergents we have

$$48 \times 2 - 5 \times 19 = 1$$

and therefore, multiplying through by 4 and rearranging,

$$48 \times 8 - 19 \times 20 = 4.$$

This gives $x = 8, y = -20$ as one solution of the linear Diophantine equation $48x + 19y = 4$, and so the general solution is

$$x = 8 + 19t, \quad y = -20 - 48t, \quad t \in \mathbb{Z}.$$

Solution 1.9

The convergents of $[2, 1, 1, 2, 1, 1, 2]$ are

$$\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{13}{5}, \frac{18}{7}, \frac{31}{12} \text{ and } \frac{80}{31}.$$

Theorem 1.4 orders the convergents as

$$2 < \frac{5}{2} < \frac{18}{7} < \frac{80}{31} < \frac{31}{12} < \frac{13}{5} < 3,$$

which is readily confirmed by evaluating these as decimals. Our results are recorded to a maximum of three decimal places.

$$2 < 2.5 < 2.571 < 2.581 < 2.583 < 2.6 < 3.$$

To avoid approximate division note that, for example, $5 \times 31 < 12 \times 13$ from which $\frac{31}{12} < \frac{13}{5}$ follows. Each of the inequalities can be tested in this way by cross-multiplication.

Solution 1.10

For any FCF all partial quotients (with the possible exception of a_1) are positive. So $q_1 = 1$, $q_2 = a_2$ and $q_3 = a_3 a_2 + 1$ satisfy $1 = q_1 \leq q_2 < q_3$.

Heading for an induction proof, suppose that $q_k > q_{k-1} \geq 1$ for some $k \geq 3$. Then

$$q_{k+1} = a_{k+1} q_k + q_{k-1} \geq q_k + q_{k-1} \geq q_k + 1 > q_k,$$

and the induction step is established and the proof is complete.

Finally, since $q_3 \geq 2$, the above inequality $q_{k+1} \geq q_k + q_{k-1}$ shows that, from q_4 onwards, the q_ks increase in steps of at least 2 and as $q_4 \geq 3$, $q_k \geq k$, for $k \geq 5$ follows. (This could be proved slightly more formally by using Mathematical Induction.)

In fact, the 'smallest' sequence of denominators occurs when every partial quotient (after a_1) takes its minimum value of 1, and then the sequence of denominators is 1, 1, 2, 3, 5, 8, ..., which is the Fibonacci sequence.

Solution 2.1

The convergents of the ICF $[1, 1, 1, 1, \ldots]$ begin $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}$.

The error in $C_5 = \frac{8}{5}$ is bounded above by $\frac{1}{5 \times 8} = 0.025$.

The error in $C_6 = \frac{13}{8}$ is bounded above by $\frac{1}{8 \times 13} = 0.009615$ (approx).

The actual error in C_5 is approximately $1.61803 - 1.6 = 0.01803$.

The actual error in C_6 is approximately $1.61803 - 1.625 = -0.00697$.

Solution 2.2

If we put $x = [3, 1, 3, 3, 1, 3, 3, 1, 3, \ldots]$ then, as in Example 2.1, $x = [3, 1, 3, x]$. The convergents of $[3, 1, 3, x]$ are

$$\frac{3}{1}, \frac{4}{1}, \frac{15}{4} \text{ and } \frac{15x+4}{4x+1},$$

and, as the final convergent is x itself,

$$x = \frac{15x+4}{4x+1}, \text{ that is } 4x^2 - 14x - 4 = 0.$$

As $x > a_1 = 3$, it is the positive root of this quadratic equation and so $x = \dfrac{7+\sqrt{65}}{4}$.

Solution 2.3

Putting $x = [2, 2, 2, 2, \ldots]$ we have $x = [2, x]$. The convergents of $[2, x]$ are $\dfrac{2}{1}$ and $\dfrac{2x+1}{x}$. As the second of these is equal to x itself we reach the quadratic $x^2 - 2x - 1 = 0$ which has roots $x = 1 \pm \sqrt{2}$. As $x > a_1 = 2$ it is the positive root of this quadratic equation and so

$$[2, 2, 2, 2, \ldots] = x = 1 + \sqrt{2}.$$

Now notice that we can add an integer to the value of an ICF by adding it to the leading partial quotient a_1, that is

$$[a_1, a_2, a_3, \ldots] + n = [a_1 + n, a_2, a_3, \ldots].$$

Here, adding -1 to both sides,

$$[1, 2, 2, 2, \ldots] = \sqrt{2},$$

which gives the ICF of $\sqrt{2}$. Its convergents begin:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}.$$

Now $\sqrt{2}$ lies between each pair of successive convergents. So, for example, it lies between $\dfrac{41}{29}$ and $\dfrac{99}{70}$, whose difference in size is $\dfrac{1}{29 \times 70}$. Hence $\dfrac{41}{29}$ lies within $\dfrac{1}{29 \times 70}$ $\left(\text{which is less than} \dfrac{1}{1000}\right)$ of $\sqrt{2}$.

Solution 2.4

(a) Let $x = [\langle 1, 1, 1, 4\rangle] = [1, 1, 1, 4, x]$. The convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{14}{9} \text{ and } \frac{14x+3}{9x+2} = x.$$

The resulting quadratic equation is $9x^2 - 12x - 3 = 0$ which, after cancelling the common factor 3, becomes $3x^2 - 4x - 1 = 0$. This has the positive root $x = \dfrac{2+\sqrt{7}}{3}$. So

$$[2, \langle 1, 1, 1, 4\rangle] = [2, x] = 2 + \frac{1}{x} = 2 + \frac{3}{2+\sqrt{7}}$$

$$= \frac{7+2\sqrt{7}}{2+\sqrt{7}} = \frac{\sqrt{7} \times (\sqrt{7}+2)}{2+\sqrt{7}} = \sqrt{7}.$$

(b) Let $x = [\langle 3\rangle] = [3, x]$. Then $x = 3 + \dfrac{1}{x}$, giving the quadratic $x^2 - 3x - 1 = 0$ which has positive root $x = \dfrac{3+\sqrt{13}}{2}$. Now

$$[2, 2, \langle 3\rangle] = [2, 2, x]$$

has convergents $\frac{2}{1}$, $\frac{5}{2}$ and $\frac{5x+2}{2x+1}$, and therefore

$$[2,2,\langle 3\rangle] = \frac{5x+2}{2x+1} = \frac{5\times\frac{3+\sqrt{13}}{2}+2}{2\times\frac{3+\sqrt{13}}{2}+1} = \frac{19+5\sqrt{13}}{8+2\sqrt{13}},$$

and this, when simplified by multiplying top and bottom by $8-2\sqrt{13}$, gives

$$[2,2,\langle 3\rangle] = \frac{11+\sqrt{13}}{6}.$$

Solution 3.1

(a) $x_1 = \sqrt{3}$ lies between 1 and 2 and so $\text{int}(\sqrt{3}) = 1$. Therefore

$\sqrt{3} = 1+(\sqrt{3}-1)$, giving $a_1 = 1$ and $x_2 = \dfrac{1}{\sqrt{3}-1}$.

Repeating for x_2 we have

$$\frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2} = 1+\frac{\sqrt{3}-1}{2}, \text{ giving } a_2 = 1 \text{ and } x_3 = \frac{2}{\sqrt{3}-1}.$$

From x_3:

$$\frac{2}{\sqrt{3}-1} = \sqrt{3}+1 = 2+(\sqrt{3}-1), \text{ giving } a_3 = 2 \text{ and } x_4 = \frac{1}{\sqrt{3}-1}.$$

As $x_4 = x_2$ we are now in a repeating cycle with $a_4 = a_2$, $a_5 = a_3$, $a_6 = a_4$, etc. So $\sqrt{3} = [1,\langle 1,2\rangle]$.

In each of the last two calculations we multiplied top and bottom by $\sqrt{3}+1$ and then pulled out the integer part.

(b) $x_1 = 2+\sqrt{7} = 4+(\sqrt{7}-2)$; $\quad a_1 = 4, \quad x_2 = \dfrac{1}{\sqrt{7}-2}$

$$x_2 = \frac{1}{\sqrt{7}-2}\times\frac{\sqrt{7}+2}{\sqrt{7}+2} = \frac{\sqrt{7}+2}{3} = 1+\frac{\sqrt{7}-1}{3}; \quad a_2 = 1, \quad x_3 = \frac{3}{\sqrt{7}-1}$$

$$x_3 = \frac{3}{\sqrt{7}-1}\times\frac{\sqrt{7}+1}{\sqrt{7}+1} = \frac{\sqrt{7}+1}{2} = 1+\frac{\sqrt{7}-1}{2}; \quad a_3 = 1, \quad x_4 = \frac{2}{\sqrt{7}-1}$$

$$x_4 = \frac{2}{\sqrt{7}-1}\times\frac{\sqrt{7}+1}{\sqrt{7}+1} = \frac{\sqrt{7}+1}{3} = 1+\frac{\sqrt{7}-2}{3}; \quad a_4 = 1, \quad x_5 = \frac{3}{\sqrt{7}-2}$$

$$x_5 = \frac{3}{\sqrt{7}-2}\times\frac{\sqrt{7}+2}{\sqrt{7}+2} = \sqrt{7}+2 = x_1$$

Hence $a_5 = a_1$, $a_6 = a_2$, $a_7 = a_3$, etc.

Therefore $2+\sqrt{7} = [\langle 4,1,1,1\rangle]$.

Solution 3.2

For $n \geq 1$ the integer part of $\sqrt{n^2+2}$ is n and so we write

$$\sqrt{n^2+2} = n+(\sqrt{n^2+2}-n),$$

giving $a_1 = n$ and $x_2 = \dfrac{1}{\sqrt{n^2+2}-n}$. This expression for x_2 is simplified by multiplying top and bottom by $\sqrt{n^2+2}+n$:

$$\frac{1}{\sqrt{n^2+2}-n} = \frac{\sqrt{n^2+2}+n}{2} = n+\frac{\sqrt{n^2+2}-n}{2},$$

giving $a_2 = n$ and $x_3 = \dfrac{2}{\sqrt{n^2+2}-n}$.

$$\frac{2}{\sqrt{n^2+2}-n} = \sqrt{n^2+2}+n = 2n+(\sqrt{n^2+2}-n),$$

giving $a_3 = 2n$ and $x_4 = x_2$.

This pair of equations now cycle and so $\sqrt{n^2+2} = [n, \langle n, 2n\rangle]$.

Hence we have:

$$\sqrt{3} = [1, \langle 1, 2\rangle];$$
$$\sqrt{6} = [2, \langle 2, 4\rangle];$$
$$\sqrt{11} = [3, \langle 3, 6\rangle];$$
$$\sqrt{18} = [4, \langle 4, 8\rangle].$$

Solution 3.3

(a) From the evidence presented in Table 3.1 you might have made some of the following observations:

- the ICF of $\sqrt{n}$ is periodic, with cycle starting at a_2;
- the final partial quotient in the cycle is $2a_1$;
- leaving aside the final term $2a_1$, what remains of the cycle is symmetric (palindromic), reading the same forwards and backwards;
- the palindromic part sometimes has even length, sometimes odd, and in some cases (e.g $\sqrt{2}$ and $\sqrt{5}$) is non-existent.

(b) The effect is as follows.

Adding a_1, which is equal to the integer part of $\sqrt{n}$, gives a first partial quotient of $2a_1$, and this can be absorbed into the cycle leaving a purely periodic ICF. For example,

$$\sqrt{7} + \text{int}(\sqrt{7}) = [\langle 4, 1, 1, 1\rangle].$$

(c) Collecting all this together we might make the conjecture that the ICF of $\sqrt{n}$ takes the form

$$\sqrt{n} = [a_1, \langle a_2, a_3, \ldots a_3, a_2, 2a_1\rangle].$$

Solution 4.1

The convergents of $\sqrt{5}$ are

$$\frac{2}{1}, \frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \ldots.$$

The convergent $\dfrac{38}{17}$ satisfies

$$\frac{1}{2 \times 17 \times 72} < \left| x - \frac{38}{17} \right| < \frac{1}{17 \times 72}.$$

That is, working to five decimal places

$$0.00041 < \left| x - \frac{38}{17} \right| < 0.00082.$$

Solution 4.2

The convergents of $\sqrt{2}$ are

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots.$$

The convergent $\dfrac{99}{70}$ satisfies

$$\left| x - \frac{99}{70} \right| < \frac{1}{70 \times 169} < \frac{1}{10\,000} = 0.0001,$$

and so has the required accuracy. It is the first convergent with this property because the previous convergent $\dfrac{41}{29}$ satisfies

$$\left| x - \frac{41}{29} \right| > \frac{1}{2 \times 29 \times 70} > \frac{1}{10\,000} = 0.0001.$$

Solution 4.3

The convergents of x are

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{30}{43}, \frac{157}{225}, \frac{972}{1393}, \ldots .$$

The convergent $\dfrac{157}{225}$ satisfies

$$\left|x - \frac{157}{225}\right| < \frac{1}{225 \times 1393} < 0.000004,$$

and so has the required accuracy. It is the first convergent with this property because the previous one, $\dfrac{30}{43}$, satisfies

$$\left|x - \frac{30}{43}\right| > \frac{1}{2 \times 43 \times 225} > 0.00005.$$

Let $\dfrac{r}{s}$ be a rational different from $\dfrac{30}{43}$ with $1 \leq s \leq 100$. Then, since $43r - 30s$ is a non-zero integer, we have

$$\left|\frac{r}{s} - \frac{30}{43}\right| = \frac{|43r - 30s|}{43s} \geq \frac{1}{43s} \geq \frac{1}{43 \times 100}.$$

But

$$\left|x - \frac{30}{43}\right| < \frac{1}{43 \times 225},$$

and so the triangle inequality gives

$$\begin{aligned}\left|x - \frac{r}{s}\right| &> \frac{1}{43 \times 100} - \frac{1}{43 \times 225} = \frac{125}{43 \times 100 \times 225} \\ &> \frac{1}{43 \times 225} > \left|x - \frac{30}{43}\right|,\end{aligned}$$

and so $\dfrac{30}{43}$ is a better approximation than $\dfrac{r}{s}$.

Solution 4.4

By Theorem 4.3, a sufficient condition for $\dfrac{17}{16}$ to be a convergent of $\dfrac{2+\sqrt{11}}{5}$ is that $\left|\dfrac{2+\sqrt{11}}{5} - \dfrac{17}{16}\right| < \dfrac{1}{2 \times 16^2}$.

This can be confirmed by use of a calculator or

$$\begin{aligned}\left|\frac{2+\sqrt{11}}{5} - \frac{17}{16}\right| &= \frac{|-53 + 16\sqrt{11}|}{80} \\ &= \frac{|(-53 + 16\sqrt{11})(-53 - 16\sqrt{11})|}{80 \times |(-53 - 16\sqrt{11})|} \\ &= \frac{7}{80 \times (53 + 16\sqrt{11})} \\ &< \frac{7}{8000} < \frac{1}{2 \times 16^2}.\end{aligned}$$

$53^2 = 2809$
$11 \times 16^2 = 2816$

$53 + 16\sqrt{11} > 53 + 16 \times 3 > 100$

SOLUTIONS TO ADDITIONAL EXERCISES

Section 1

1 (a) $23 = 0 \times 54 + 23$

$54 = 2 \times 23 + 8$

$23 = 2 \times 8 + 7$

$8 = 1 \times 7 + 1$

$7 = 7 \times 1$

So $\dfrac{23}{54} = [0, 2, 2, 1, 7]$.

(b) $117 = 6 \times 17 + 15$

$17 = 1 \times 15 + 2$

$15 = 7 \times 2 + 1$

$2 = 2 \times 1$

So $\dfrac{117}{17} = [6, 1, 7, 2]$.

(c) $-37 = (-2) \times 24 + 11$

$24 = 2 \times 11 + 2$

$11 = 5 \times 2 + 1$

$2 = 2 \times 1$

So $\dfrac{-37}{24} = [-2, 2, 5, 2]$.

(d) We treat 3.142 as $\dfrac{3142}{1000}$.

$3142 = 3 \times 1000 + 142$

$1000 = 7 \times 142 + 6$

$142 = 23 \times 6 + 4$

$6 = 1 \times 4 + 2$

$4 = 2 \times 2$

So $3.142 = [3, 7, 23, 1, 2]$.

Notice that this works without reducing the rational numbers to lowest terms. We leave you to ponder why. Generally we consider our rational numbers to be in lowest terms.

2 (a) If $x = [a_1, a_2, a_3, \ldots a_n] > 1$ then

$$\frac{1}{x} = 0 + \frac{1}{x} = 0 + \frac{1}{[a_1, a_2, a_3, \ldots a_n]} = [0, a_1, a_2, a_3, \ldots a_n].$$

Note that a_1 is not the leading partial quotient in this FCF and so it is essential that $a_1 \geq 1$, a fact which is guaranteed by $x > 1$.

For $0 < x < 1$ we have $\dfrac{1}{x} > 1$, and so we can apply the result just obtained to $\dfrac{1}{x}$. If $\dfrac{1}{x} = [b_1, b_2, b_3, \ldots b_n]$ then the FCF of its reciprocal (noting that $\dfrac{1}{1/x} = x$) is given by $x = [0, b_1, b_2, b_3, \ldots b_n]$. In other words, if $0 < x < 1$ then its FCF will have leading partial quotient 0, say $x = [0, a_1, a_2, \ldots a_n]$ and then $\dfrac{1}{x} = [a_1, a_2, \ldots a_n]$.

(b) If $x = [-1, a_1, a_2, \ldots a_n]$ then $1 + x = [0, a_1, a_2, \ldots a_n]$. By the result of part (a) the reciprocal of $1 + x$ has FCF $[a_1, a_2, \ldots a_n]$.

3 (a) $[2,2,4,2,4,2]$ has convergents

$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}, \frac{218}{89} \text{ and } \frac{485}{198}.$$

According to Theorem 1.4 we have the inequalities

$$\frac{2}{1} < \frac{22}{9} < \frac{218}{89} < \frac{485}{198} < \frac{49}{20} < \frac{5}{2},$$

each of which is readily verified.

(b) $[1,2,3,4,5,6,7]$ has convergents

$$\frac{1}{1}, \frac{3}{2}, \frac{10}{7}, \frac{43}{30}, \frac{225}{157}, \frac{1393}{972} \text{ and } \frac{9976}{6961}.$$

The inequalities given by Theorem 1.4 are

$$1 < \frac{10}{7} < \frac{225}{157} < \frac{9976}{6961} < \frac{1393}{972} < \frac{43}{30} < \frac{3}{2},$$

each of which is readily verified.

4 (a) The FCF of $\frac{91}{17}$ is $[5,2,1,5]$ and it has convergents

$$\frac{5}{1}, \frac{11}{2}, \frac{16}{3} \text{ and } \frac{91}{17}.$$

From the final pair of convergents $3 \times 91 - 16 \times 17 = 1$, which shows that $x = -16$, $y = 3$ is one solution of $17x + 91y = 1$. The general solution is therefore given by

$$x = -16 + 91t, \quad y = 3 - 17t, \quad t \in \mathbb{Z}.$$

(b) The FCF of $\frac{219}{16}$ is $[13,1,2,5]$ and has convergents

$$\frac{13}{1}, \frac{14}{1}, \frac{41}{3} \text{ and } \frac{219}{16}.$$

From the final two convergents we have $3 \times 219 - 16 \times 41 = 1$.

Multiplying through by 17 gives $x = (-41) \times 17 = -697$ and $y = 3 \times 17 = 51$ as one solution of $16x + 219y = 17$, and the general solution is given by

$$x = -697 + 219t, \quad y = 51 - 16t, \quad t \in \mathbb{Z}.$$

Alternatively since $t = 3$ gives the solution $x = -40$, $y = 3$,

$$x = -40 + 219t, \quad y = 3 - 16t, \quad t \in \mathbb{Z}.$$

Section 2

1 (a) Let $x = [\langle 5,1,3 \rangle]$. Then $x = [5,1,3,x]$ and this has convergents

$$\frac{5}{1}, \frac{6}{1}, \frac{23}{4} \text{ and } \frac{23x+6}{4x+1}.$$

As the final convergent is equal to x we have the quadratic equation

$$4x^2 - 22x - 6 = 0.$$

The positive root of this equation is $x = \dfrac{11 + \sqrt{145}}{4}$.

(b) $[0, \langle 5, 1, 3\rangle]$ has value $\dfrac{1}{x}$, where x is as in part (a). That is

$$[0, \langle 5, 1, 3\rangle] = \frac{4}{11 + \sqrt{145}},$$

and this can be simplified by multiplying top and bottom by $11 - \sqrt{145}$:

$$[0, \langle 5, 1, 3\rangle] = \frac{4(11 - \sqrt{145})}{-24} = \frac{\sqrt{145} - 11}{6}.$$

(c) $[1, 2, \langle 5, 1, 3\rangle] = [1, 2, x]$ and has convergents $\dfrac{1}{1}$, $\dfrac{3}{2}$ and $\dfrac{3x+1}{2x+1}$. So, substituting for x from (a),

$$[1, 2, \langle 5, 1, 3\rangle] = \frac{37 + 3\sqrt{145}}{26 + 2\sqrt{145}},$$

which can be simplified to $\dfrac{23 + \sqrt{145}}{24}$.

2 (a) Taking $k = 2$ we have $\dfrac{e-1}{e+1} = [0, 2, 6, 10, 14, 18, \ldots]$. The convergents begin

$$\frac{0}{1}, \frac{1}{2}, \frac{6}{13}, \frac{61}{132}, \frac{860}{1861}, \ldots.$$

The convergent $\dfrac{61}{132}$ is accurate to within $\dfrac{1}{132 \times 1861}$ which is less than 5×10^{-6} and guarantees correctness to four decimal places.

(b) Taking $k = 1$ we have $\dfrac{e^2-1}{e^2+1} = [0, 1, 3, 5, 7, 9, \ldots]$. The convergents are

$$\frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{16}{21}, \frac{115}{151}, \frac{1051}{1380}, \ldots.$$

The convergent $\dfrac{115}{151}$ is accurate to within $\dfrac{1}{151 \times 1380}$ which is less than 5×10^{-6} and guarantees correctness to four decimal places.

Section 3

1 (a)
$$x_1 = \sqrt{11} = 3 + (\sqrt{11} - 3); \qquad a_1 = 3, \qquad x_2 = \frac{1}{\sqrt{11} - 3}$$
$$\frac{1}{\sqrt{11} - 3} = \frac{\sqrt{11} + 3}{2} = 3 + \frac{\sqrt{11} - 3}{2}; \qquad a_2 = 3, \qquad x_3 = \frac{2}{\sqrt{11} - 3}$$
$$\frac{2}{\sqrt{11} - 3} = \sqrt{11} + 3 = 6 + (\sqrt{11} - 3); \qquad a_3 = 6, \qquad x_4 = \frac{1}{\sqrt{11} - 3}$$

As $x_4 = x_2$ we are now in a cycle with the second and third equations recurring indefinitely, so $\sqrt{11} = [3, \langle 3, 6\rangle]$.

(b)
$$x_1 = \frac{1 + \sqrt{13}}{2} = 2 + \frac{\sqrt{13} - 3}{2}; \qquad a_1 = 2, \qquad x_2 = \frac{2}{\sqrt{13} - 3}$$
$$\frac{2}{\sqrt{13} - 3} = \frac{\sqrt{13} + 3}{2} = 3 + \frac{\sqrt{13} - 3}{2}; \qquad a_2 = 3, \qquad x_3 = \frac{2}{\sqrt{13} - 3}$$

As $x_3 = x_2$ we are immediately into a cycle and, after the initial partial quotient of 2, the partial quotient 3 recurs indefinitely.

Hence $\dfrac{1 + \sqrt{13}}{2} = [2, \langle 3\rangle]$.

(c) $x_1 = \dfrac{4+\sqrt{21}}{5} = 1 + \dfrac{\sqrt{21}-1}{5}$; $\quad a_1 = 1, \quad x_2 = \dfrac{5}{\sqrt{21}-1}$

$\dfrac{5}{\sqrt{21}-1} = \dfrac{\sqrt{21}+1}{4} = 1 + \dfrac{\sqrt{21}-3}{4}$; $\quad a_2 = 1, \quad x_3 = \dfrac{4}{\sqrt{21}-3}$

$\dfrac{4}{\sqrt{21}-3} = \dfrac{\sqrt{21}+3}{3} = 2 + \dfrac{\sqrt{21}-3}{3}$; $\quad a_3 = 2, \quad x_4 = \dfrac{3}{\sqrt{21}-3}$

$\dfrac{3}{\sqrt{21}-3} = \dfrac{\sqrt{21}+3}{4} = 1 + \dfrac{\sqrt{21}-1}{4}$; $\quad a_4 = 1, \quad x_5 = \dfrac{4}{\sqrt{21}-1}$

$\dfrac{4}{\sqrt{21}-1} = \dfrac{\sqrt{21}+1}{5} = 1 + \dfrac{\sqrt{21}-4}{5}$; $\quad a_5 = 1, \quad x_6 = \dfrac{5}{\sqrt{21}-4}$

$\dfrac{5}{\sqrt{21}-4} = \sqrt{21}+4 = 8 + (\sqrt{21}-4)$; $\quad a_6 = 8, \quad x_7 = \dfrac{1}{\sqrt{21}-4}$

$\dfrac{1}{\sqrt{21}-4} = \dfrac{\sqrt{21}+4}{5} = 1 + \dfrac{\sqrt{21}-1}{5}$; $\quad a_7 = 1, \quad x_8 = \dfrac{5}{\sqrt{21}-1}$

Now $x_8 = x_2$ and we are in a cycle, with the sequence $1, 2, 1, 1, 8, 1$ of partial quotients recurring. So

$$\frac{4+\sqrt{21}}{5} = [1, \langle 1, 2, 1, 1, 8, 1\rangle].$$

But notice that this gives

$$\frac{4+\sqrt{21}}{5} = [\langle 1, 1, 2, 1, 1, 8\rangle].$$

In fact, although it was not immediately obvious, $x_7 = x_1$ and we were in a cycle at the point of calculating x_7.

2 Since $0 < k \leq 2n$ we have $n^2 < n^2 + k < n^2 + 2n + 1 = (n+1)^2$. Therefore the integer part of $\sqrt{n^2+k}$ is n, and so

$$\sqrt{n^2+k} = n + (\sqrt{n^2+k} - n); \quad a_1 = n, \; x_2 = \frac{1}{\sqrt{n^2+k}-n},$$

$$\frac{1}{\sqrt{n^2+k}-n} = \frac{\sqrt{n^2+k}+n}{k}.$$

The integer part of $\sqrt{n^2+k}+n$ is $2n$, and as k divides $2n$ we have

$$\frac{1}{\sqrt{n^2+k}-n} = \frac{2n}{k} + \frac{\sqrt{n^2+k}-n}{k}; \quad a_2 = \frac{2n}{k}, \; x_3 = \frac{k}{\sqrt{n^2+k}-n},$$

$$\frac{k}{\sqrt{n^2+k}-n} = \sqrt{n^2+k}+n = 2n + (\sqrt{n^2+k}-n); \quad a_3 = 2n,$$

$$x_4 = \frac{1}{\sqrt{n^2+k}-n}.$$

As $x_4 = x_2$ we are in a cycle with the pair of partial quotients $\dfrac{2n}{k}, 2n$ repeating indefinitely. So

$$\sqrt{n^2+k} = \left[n, \left\langle \frac{2n}{k}, 2n \right\rangle\right].$$

(a) Take $n = 3$ and $k = 3$. Then $\sqrt{12} = [3, \langle 2, 6\rangle]$.

(b) Take $n = 4$ and $k = 8$. Then $\sqrt{24} = [4, \langle 1, 8\rangle]$.

3 For $n \geq 2$ we have

$$(n-1)^2 = n^2 - 2n + 1 < n^2 - n < n^2$$

and so $\sqrt{n^2-n}$ lies strictly between $n-1$ and n. Therefore

$$x_1 = \sqrt{n^2-n} = n - 1 + (\sqrt{n^2-n} - (n-1)), \quad \text{giving } a_1 = n-1.$$

Continuing the Continued Fraction Algorithm:

$$x_2 = \frac{1}{\sqrt{n^2-n} - (n-1)} = \frac{\sqrt{n^2-n} + (n-1)}{(n^2-n) - (n-1)^2} = \frac{\sqrt{n^2-n} + (n-1)}{n-1}.$$

As the integer part of $\sqrt{n^2-n}$ is $n-1$ the integer part of $\dfrac{\sqrt{n^2-n}+n-1}{n-1}$ is 2 and so

$$x_2 = 2 + \frac{\sqrt{n^2-n} - (n-1)}{n-1}, \quad \text{giving } a_2 = 2.$$

At the next stage we look at

$$x_3 = \frac{n-1}{\sqrt{n^2-n} - (n-1)} = \sqrt{n^2-n} + (n-1).$$

As this has integer part $2(n-1)$,

$$x_3 = 2(n-1) + (\sqrt{n^2-n} - (n-1)), \quad \text{giving } a_3 = 2(n-1).$$

And there is good news. The fractional part, $\sqrt{n^2-n} - (n-1)$, is the same as the fractional part of x_1, so we are in a cycle and can conclude that

$$\sqrt{n^2-n} = [n-1, \langle 2, 2n-2\rangle].$$

For example:

$n = 2$ gives $\sqrt{2} = [1, \langle 2, 2\rangle] = [1, \langle 2\rangle]$;
$n = 3$ gives $\sqrt{6} = [2, \langle 2, 4\rangle]$;
$n = 4$ gives $\sqrt{12} = [3, \langle 2, 6\rangle]$.

Section 4

1 (a) The convergents of $x = [0, 2, 4, 6, 8, 10, \ldots]$ are

$$\frac{0}{1}, \frac{1}{2}, \frac{4}{9}, \frac{25}{56}, \frac{204}{457}, \frac{2065}{4626}, \ldots.$$

The convergent $\dfrac{25}{56}$ satisfies $\left|x - \dfrac{25}{56}\right| < \dfrac{1}{56 \times 457}$ and is known to be the best rational approximation amongst rationals with denominator not exceeding 56. Now suppose that $\dfrac{a}{b}$ is any rational with $b \leq 100$ and different from $\dfrac{25}{56}$. Then

$$\left|\frac{a}{b} - \frac{25}{56}\right| = \frac{|56a - 25b|}{56b} \geq \frac{1}{56b} \geq \frac{1}{56 \times 100}.$$

It follows (from the triangle inequality) that

$$\begin{aligned}\left|\frac{a}{b} - x\right| &\geq \left|\frac{a}{b} - \frac{25}{56}\right| - \left|x - \frac{25}{56}\right| \\ &> \frac{1}{56 \times 100} - \frac{1}{56 \times 457} = \frac{357}{56 \times 100 \times 457} > \frac{1}{56 \times 457},\end{aligned}$$

and therefore $\dfrac{25}{56}$ is a better approximation than $\dfrac{a}{b}$.

(b) In the same vein, the convergent $\dfrac{204}{457}$ satisfies

$$\left|x - \frac{204}{457}\right| < \frac{1}{457 \times 4626}.$$

So if $\dfrac{a}{b}$ is a rational number different from $\dfrac{204}{457}$ and with $b < 1000$ then

$$\left|\frac{a}{b} - \frac{204}{457}\right| = \left|\frac{457a - 204b}{457b}\right| \geq \frac{1}{457b} \geq \frac{1}{457 \times 1000},$$

and

$$\left|\frac{a}{b} - x\right| > \frac{1}{457 \times 1000} - \frac{1}{457 \times 4626} = \frac{3626}{457 \times 4626 \times 1000} > \frac{1}{457 \times 4626}.$$

Therefore, $\dfrac{204}{457}$ is the best approximation amongst rationals with denominator not exceeding 1000.

2 Since the convergents of x are alternately greater than and smaller than x,

$$\left|x - \frac{p_{n+1}}{q_{n+1}}\right| + \left|x - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{|p_{n+1}q_n - p_nq_{n+1}|}{q_nq_{n+1}} = \frac{1}{q_nq_{n+1}}.$$

So if neither convergent satisfies the given property

$$\frac{1}{q_nq_{n+1}} \geq \frac{1}{2q_{n+1}^2} + \frac{1}{2q_n^2}.$$

This can be rearranged as $(q_{n+1} - q_n)^2 \leq 0$. As a square cannot be negative and $q_{n+1} > q_n$ except for the possibility of $q_1 = q_2 = 1$, this gives the required contradiction for all except this exceptional case. However, when $q_1 = q_2 = 1$ the first two convergents of x will be $\text{int}(x)$ and $\text{int}(x) + 1$ and the result makes the modest, and true, claim that one of these lies within $\dfrac{1}{2}$ of the true value.

3 (a) Since $\sqrt{5} > 2$, $\left|x - \dfrac{a}{b}\right| < \dfrac{1}{\sqrt{5}b^2} < \dfrac{1}{2b^2}$, and Theorem 4.3 guarantees that $\dfrac{a}{b}$ is a convergent of x.

(b) $\sqrt{3} = [1, \langle 1, 2\rangle]$ and has convergents

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \ldots.$$

It can be checked by calculator that $\left|\sqrt{3} - 1\right| > \dfrac{1}{\sqrt{5}}$, $\left|\sqrt{3} - 2\right| < \dfrac{1}{\sqrt{5}}$, $\left|\sqrt{3} - \dfrac{5}{3}\right| > \dfrac{1}{9\sqrt{5}}$ and $\left|\sqrt{3} - \dfrac{7}{4}\right| < \dfrac{1}{16\sqrt{5}}$, so the convergents $\dfrac{1}{1}$ and $\dfrac{5}{3}$ fail to satisfy the inequality while the convergents $\dfrac{2}{1}$ and $\dfrac{7}{4}$ do satisfy it.

INDEX

C_k 11
$[a_1, a_2, a_3, \ldots]$ 20
$[a_1, a_2, a_3, \ldots, a_n]$ 7
$[\langle a_1, a_2, a_3, \ldots, a_n \rangle]$ 24
frac(x) 27
int(x) 27
$\{C_n\}$ 20

accuracy of convergents 33

best approximations 36

Continued Fraction Algorithm 26, 27
convergent 11
cycle of a periodic ICF 24

denominators 12

equivalence of FCFs and rational numbers 9
even convergents 17

FCF 8
finite continued fraction 7
fractional part 27

golden ratio 5

ICF 20
infinite continued fraction 20
irrationality of ICFs 26

numerators 12

odd convergents 17

partial quotients 7
periodic 24
properties of the convergents 16
purely periodic 24

quadratic irrationals 30

rational approximations 32
rational number 9

simple continued fraction 7
simple infinite continued fraction 20

value of a continued fraction 7
value of an infinite continued fraction 21